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Improved Estimation of Population Variance Incorporating Auxiliary Information in the Presence of Measurement Errors

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ABSTRACT

This study proposes a new class of estimators for using two auxiliary variables under classical additive measurement error a common issue in survey data. While traditional estimators assume error post-stratified population variance -free auxiliary information, this work explicitly accounts for measurement error, deriving bias and mean squared error (MSE) up to the first order of approximation. The proposed estimators incorporate known population parameters (means, coefficients of variation, correlation) and are evaluated using two benchmark datasets: Murthy (1967) and Kadilar & Cingi (2006). Results in Tables 1 and 2 shows that ignoring measurement error inflates MSE, leading to overestimation or underestimation of variance. Efficiency comparisons confirm that existing and proposed estimators perform poorly under error contamination. The study highlights the critical impact of data quality on inference and underscores the need for error-corrected estimation methods and robust data collection practices in survey sampling.

Keywords: *Population Variance, Post-Stratified Sampling, Auxiliary Variables, Measurement Error, Mean Squared Error (MSE), Robust Estimation, Survey Sampling, Efficiency Comparison*

INTRODUCTION

The accurate estimation of population variance is a fundamental task in statistical inference, with wide-ranging applications in survey sampling, quality control, financial risk assessment, and socio-economic research. While traditional estimators rely solely on the study variable, substantial efficiency improvements can be achieved by incorporating auxiliary information—variables that are either highly correlated with the study variable or readily available at low cost (Cochran, 1977; Singh & Karan, 2021). Over the decades, numerous ratio, product, and regression-type estimators have been developed to estimate population variance by leveraging auxiliary variables (Isaki, 1983; Kadilar & Cingi, 2006). These estimators exploit the relationship between the study variable and one or more auxiliary variables to reduce bias and mean squared error (MSE), thereby enhancing precision. However, most classical variance estimators assume error-free measurement of both the study and auxiliary variables. In practice, this assumption is often violated due to data collection errors, respondent misreporting, instrument inaccuracy, or rounding errors. The presence of measurement error in auxiliary variables can severely distort the performance of conventional estimators, leading to biased estimates and inflated MSE (Fuller, 2009; Lohr, 2022). Recent studies have emphasized the importance of accounting for measurement error in estimation procedures.

For instance, Sahoo & Sahoo (2020) demonstrated that ignoring measurement error structures in auxiliary variables leads to substantial efficiency loss in ratio-type variance estimators. Similarly, Singh and Karan (2021) proposed modified ratio estimators under additive measurement error models, showing improved robustness in the presence of noise. More recently, Khan et al. (2022) introduced a class of exponential-type variance estimators that incorporate auxiliary information under both multiplicative and additive measurement error structures. Their simulation and empirical results confirmed that accounting for measurement error significantly improves estimator performance, particularly in large-scale surveys and administrative data settings. Further advancements include the use of robust measures, such as the median, deciles mean, and tri-mean of auxiliary variables, to mitigate the impact of outliers and measurement errors (Riaz et al., 2023). These non-conventional location measures offer greater resistance to contamination, making them ideal for real-world data where errors are common. Despite these developments, a gap remains in the literature regarding the efficient and robust estimation of population variance under realistic measurement error conditions, especially when auxiliary variables are subject to classical additive error or non-differential misclassification. This study contributes to the existing body of knowledge by:

- Proposing a new class of improved variance estimators that utilize auxiliary information through conventional and non-conventional measures (e.g., median, Hodges-Lehmann estimator, and mid-range).
- Investigating the effect of measurement error in auxiliary variables on the bias and MSE of the proposed estimators.
- Deriving analytical expressions for bias and mean squared error up to the first order of approximation.
- Conducting simulation studies and empirical applications to compare the performance of the proposed estimators with existing ones under various error structures.

A large body of literature has focused on developing ratio, product, regression, and exponential-type estimators for population variance using auxiliary variables (Singh & Karan, 2021; Khan et al., 2022). These estimators leverage the relationship between the study variable and the auxiliary variable to minimize mean squared error (MSE) and enhance estimation accuracy. However, the vast majority of these methods assume that the auxiliary variable is measured without error an assumption that is frequently violated in real-world applications. In practice, measurement error is a common issue in data collection, arising from various sources, including respondent misreporting, instrument inaccuracy, data entry errors, rounding, or proxy reporting (Fuller, 2009; Buonaccorsi, 2023). When auxiliary variables are contaminated by measurement error, traditional variance estimators suffer from increased bias and inflated MSE, leading to misleading inferences and reduced efficiency (Sahoo & Sahoo, 2020; Singh & Karan, 2021). Recent studies have begun to address this limitation. For instance, Khan et al. (2022) proposed exponential-type variance estimators under additive and multiplicative measurement error models, demonstrating improved performance over conventional estimators. Similarly, Riaz et al. (2023) investigated the application of robust location measures (e.g., median, decile mean, tri-mean) in auxiliary variables to mitigate the effects of outliers and measurement errors, demonstrating enhanced resilience in contaminated environments.

Considering finite population involving N units, and further divided into L non-coinciding strata with h^{th} stratum consisting of N_h units, where $h = 1,2,3.......,L$ so that $\sum_{h=1}^{L} N_h = N$. Let n being the sample size drawn from a population by using simple random sampling without replacement scheme such that $\sum_{h=1}^{L} n_h = n$. Also it is considered that n is large therefore the possibility of n_h zero is fewer. Furthermore, y_{hi} , x_{hi} and z_{hi} are the observed values of y , x and z variables respectively, on i^{th} unit at the h^{th} stratum. Let $\bar{y}_{st} = \sum_{h=1}^{L} W_h \bar{y}_h$, $\bar{x}_{st} = \sum_{h=1}^{L} W_h \bar{x}_h$ and $\bar{z}_{st} = \sum_{h=1}^{L} W_h \bar{z}_h$ are the stratified sample means of $\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$, $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$ and $\bar{Z} = \sum_{h=1}^{L} W_h \bar{Z}_h$, respectively. Also let $\bar{y}_h = \frac{1}{n_h}\sum_{h=1}^{n_h} y_{hi}$, $\bar{x}_h = \frac{1}{n_h}\sum_{h=1}^{n_h} x_{hi}$ and $\bar{z}_h = \frac{1}{n_h}\sum_{h=1}^{n_h} z_{hi}$ are the sample means corresponding to the population means $\bar{Y}_h = \frac{1}{N_h}\sum_{h=1}^{N_h} Y_{hi}$, $\bar{X}_h = \frac{1}{N_h}\sum_{h=1}^{N_h} X_{hi}$ and $\bar{Z}_h = \frac{1}{N_h}\sum_{h=1}^{N_h} Z_{hi}$ respectively to the stratum h, where $W_h = \frac{N_h}{N}$.

Let $s_{yh}^2 = \frac{1}{n_h - 1}\sum_{i=1}^{n_h}\left(y_{hi} - \bar{y}_h\right)^2$, $s_{xh}^2 = \frac{1}{n_h - 1}\sum_{i=1}^{n_h}\left(x_{hi} - \bar{x}_h\right)^2$ and $s_{zh}^2 = \frac{1}{n_h - 1}\sum_{i=1}^{n_h}\left(z_{hi} - \bar{z}_h\right)^2$ are the sample variances corresponding to the population variance $S_{Yh}^2 = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(Y_{hi} - \bar{Y}_h\right)^2$, $S_{Xh}^2 = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(X_{hi} - \bar{X}_h\right)^2$ and $S_{Zh}^2 = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(Z_{hi} - \bar{Z}_h\right)^2$ of y_h , x_h and z_h in the h^{th} stratum. Let $c_{yh} = \frac{s_{yh}}{\bar{y}_h}$, $c_{xh} = \frac{s_{xh}}{\bar{x}_h}$ and $c_{zh} = \frac{s_{zh}}{\bar{z}_h}$ be the sample coefficient of variation corresponding to population coefficient of variation $C_{yh} = \frac{S_{yh}}{\bar{Y}_h}$, $C_{xh} = \frac{C_{xh}}{\bar{X}_h}$ and $C_{zh} = \frac{S_{zh}}{\bar{Z}_h}$ of y_h, x_h and z_h respectively. Let $r_{yxh} = \frac{s_{yxh}}{s_{yh}s_{xh}}$, $r_{zxh} = \frac{s_{zxh}}{s_{zh}s_{xh}}$ and $r_{yzh} = \frac{s_{yzh}}{s_{yh}s_{zh}}$ are sample correlation coefficients corresponding to the population correlation coefficient $\rho_{yxh} = \dfrac{S_{yxh}}{S_{yh}S_{xh}}$, $\rho_{zxh} = \dfrac{S_{zxh}}{S_{zh}S_{xh}}$ and $\rho_{yzh} = \dfrac{S_{yzh}}{S_{yh}S_{zh}}$ in the stratum h. Let $s_{yxh} = \dfrac{1}{n_h - 1}\sum_{i=1}^{n_h}\left(y_{hi} - \bar{y}_h\right)\left(x_{hi} - \bar{x}_h\right)$, $s_{xzh} = \dfrac{1}{n_h - 1}\sum_{i=1}^{n_h}\left(x_{hi} - \bar{x}_h\right)\left(z_{hi} - \bar{z}_h\right)$ and $s_{yzh} = \dfrac{1}{n_h - 1}\sum_{i=1}^{n_h}\left(y_{hi} - \bar{y}_h\right)\left(z_{hi} - \bar{z}_h\right)$ are sample covariances of their respective population covariances $S_{yxh} = \dfrac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(Y_{hi} - \bar{Y}_h\right)\left(X_{hi} - \bar{X}_h\right)$, $S_{xzh} = \dfrac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(X_{hi} - \bar{X}_h\right)\left(Z_{hi} - \bar{Z}_h\right)$ and $S_{yzh} = \dfrac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(Y_{hi} - \bar{Y}_h\right)\left(Z_{hi} - \bar{Z}_h\right)$ among their corresponding subscript in the stratum h.

Let $\left(y_h,\ x_h,\ z_h\right)$ be study and auxiliary variables defined on a finite population $U = \{U_1, U_2, \ldots\ldots., U_N\}$ of size N_h. Assume that we have given a set of n_h paired observations taken from simple random sampling procedure multiple characteristics x_h, y_h and z_h. It is considered that x_h, y_h and z_h for the j^{th} sampling unit are recorded instead of true values X_h, Y_h and Z_h. The measurement errors are defined as $u_h = \sum_{h=1}^{n}\left(z_h - Z_h\right)$ are assumed to be stochastic with zero mean and varying variance σ_{uh}^2, where $E\left(\sigma_{zh}^{2*}\right) = \sigma_{zh}^2 + \sigma_{uh}^2$. Let the error variance σ_{uh}^2 associated with z_h, then an unbiased estimator of population variance σ_{zh}^2 is given by $\hat{\sigma}_{zh}^{2*} = \sigma_{zh}^2 - \sigma_{uh}^2 > 0$.

Let

$$A'_{zh} = \left\{\varphi_h C_{zh}^2 + 2\left(1 + \frac{\sigma_{uh}^2}{\sigma_{yh}^2}\right)^2\right\},\quad A_{zh} = \left\{\gamma_{2zh} + \gamma_{2uh}\frac{\sigma_{uh}^4}{\sigma_{yh}^4} + 2\left(1 + \frac{\sigma_{uh}^2}{\sigma_{yh}^2}\right)^2\right\},\quad \gamma_{2yh} = \beta_{2h}\left(y_h\right) - 3,$$

$$\gamma_{2zh} = \beta_{2h}\left(z_h\right) - 3,\ \gamma_{2uh} = \beta_{2h}\left(u_h\right) - 3,$$

$$\lambda_h = \frac{u_{12h}\left(x_h, y_h\right)}{\sigma_{xh}\sigma_{yh}^2},\quad \lambda_{rsth} = \frac{u_{rsth}}{u_{200h}^{r/2}u_{020h}^{s/2}u_{002h}^{t/2}},\quad u_{rsth} = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(y_{hi} - \bar{Y}_h\right)^r\left(x_{hi} - \bar{X}_h\right)^s\left(z_{hi} - \bar{Z}_h\right)^t,$$

$\beta_{2h}\left(k_h\right) = \dfrac{u_{4h}\left(k_h\right)}{u_{2h}^2\left(k_h\right)}$ where $k_h = x_h, y_h, z_h$, $u_{rh}\left(k_h\right) = E\left(K_{hi} - u_{kh}\right)^r$ where $k_h = x_h, y_h, z_h$.

EXISTING ESTIMATORS

Masood and Shabbir (2015) introduced a class of ratio-type estimators for the finite population variance of the post-stratified sample mean by leveraging information from two auxiliary variables**.** Their proposed estimators demonstrated improved efficiency over conventional methods under stratified random sampling, particularly when the auxiliary variables exhibited high correlation with the study variable. However, their methodology assumed error-free measurement of auxiliary variables—a condition often violated in practice due to measurement error, non-response, or data contamination. More recent studies have addressed these limitations by incorporating robust auxiliary measures (e.g., median, decile mean) and explicitly modeling measurement error structures (Riaz et al., 2023; Khan et al., 2022). The estimators are given below,

Estimator of σ^2_{pst}***, when*** S^2_{xh} ***and*** $\bar{Z}_h$ ***are known,***

$$\hat{\sigma}^2_{P1} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} s^2_{yh,lr} a_h\left(u_h\right) \tag{1}$$

where $s^2_{yh,lr} = \hat{\sigma}^2_{yh} + \hat{\beta}_{220h}\left(S^2_{xh} - s^2_{xh}\right)$, $u_h = \frac{\bar{z}_h}{\bar{Z}_h}$ and also $a_h\left(.\right)$ is the function of u_h that is $a_h\left(1\right) = 1$.

The $Bias\left(\hat{\sigma}^2_{P1}\right)$ to the first order of approximation is given as,

$$Bias\left(\hat{\sigma}^2_{P1}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} S^2_{yh}\left[a_h\left(1\right)\left\{\lambda_{201h} - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\lambda_{021h}\right\} C_{zh} + a_{h2}\left(1\right)\frac{C_{zh}^2}{2}\right] \tag{2}$$

The $MSE\left(\hat{\sigma}^2_{P1}\right)$, to the first order of approximation, for the optimum value of

$a_h\left(1\right) = -\frac{1}{C_{zh}} S^2_{yh}\left\{\lambda_{201h} - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\lambda_{021h}\right\}$, the resulting minimum $MSE\left(\hat{\sigma}^2_{P1}\right)$, is given by

$$MSE_{\min}\left(\hat{\sigma}^2_{P1}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} S^4_{yh}\left[\left(\lambda_{400h}-1\right) - \frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)} - \left\{\frac{\lambda_{021h}\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} - \lambda_{201h}\right\}^2\right] \tag{3}$$

Estimator of σ^2_{pst}***, when*** S^2_{xh} ***and*** S^2_{zh} ***are known***

$$\hat{\sigma}^2_{Q1} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} s^2_{yh,lr} b_h\left(v_h\right) \tag{4}$$

Where $v_h = \frac{s^2_{zh}}{S^2_{zh}}$ and $b_h(.)$ is the function of v_h such that $b_h(1) = 1$.

The $Bias\left(\hat{\sigma}^2_{Q1}\right)$ to the first order of approximation is given as,

$Bias\left(\hat{\sigma}_{Q1}^{2}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} S_{yh}^2 \left[b_{h1} \left(\left(\lambda_{202h}-1\right) - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} \left(\lambda_{022h}-1\right) \right) + b_{h2} \frac{\left(\lambda_{004h}-1\right)}{2} \right]$ (5) The $MSE\left(\hat{\sigma}_{Q1}^{2}\right)$, to the first order of approximation, for the optimum value of $b_{h1}(1) = -\frac{1}{\left(\lambda_{004h}-1\right)} \left\{ \left(\lambda_{202h}-1\right) - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} \left(\lambda_{022h}-1\right) \right\}$

The $MSE\left(\hat{\sigma}_{Q1}^{2}\right)_{\min}$ is given by,

$$MSE_{\min}\left(\hat{\sigma}_{Q1}^{2}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} S_{yh}^4 \left[\left(\lambda_{400h}-1\right) - \frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)} - \frac{1}{\left(\lambda_{004h}-1\right)} \left\{ \left(\lambda_{202h}-1\right) - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} \left(\lambda_{022h}-1\right) \right\}^2 \right] \quad (6)$$

Estimator of σ_{pst}^2, when S_{xh}^2 and ρ_{xzh} are known,

$$\hat{\sigma}_{R1}^{2} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} s_{yh,lr}^2 d_h\left(\tau_h\right) \quad (7)$$

Where $\tau_h = \frac{r_{xzh}}{\rho_{xzh}}$ and $d_h(1)$ is the function of τ_h such that $d_h(1)=1$. The $Bias\left(\hat{\sigma}_{R1}^{2}\right)$ to the first order of approximation is given as,

$$\begin{aligned} Bias\left(\hat{\sigma}_{R1}^{2}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} S_{yh}^2 \Bigg[\Bigg\{ & d_{h1}(1) \left(\frac{1}{2}\left(\lambda_{220h}-1\right) + \frac{1}{2} \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} \left(\lambda_{022h}-1\right) + \frac{3}{8}\left(\lambda_{040h}-1\right) \right. \\ & - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} \left(\frac{\lambda_{031h}}{\rho_{xzh}} - 1 \right) - \frac{1}{2}\left(\lambda_{220h}-1\right) - \frac{1}{2}\left(\lambda_{202h}-1\right) - \frac{1}{2}\left(\frac{\lambda_{013h}}{\rho_{xzh}} - 1 \right) \\ & + \left(\frac{\lambda_{211h}}{\rho_{xzh}} - 1 \right) - \frac{1}{2}\left(\frac{\lambda_{031h}}{\rho_{xzh}} - 1 \right) + \frac{1}{4}\left(\lambda_{022h}-1\right) + \frac{3}{8}\left(\lambda_{004h}-1\right) \Bigg\} + \frac{d_{h2}(1)}{2} \Bigg\{ \frac{1}{2} \\ & \left(\lambda_{022h}-1\right) + \frac{1}{4}\left(\lambda_{004h}-1\right) + \left(\frac{\lambda_{022h}}{\rho_{xzh}^{2}} - 1 \right) - \left(\frac{\lambda_{031h}}{\rho_{xzh}} - 1 \right) + \frac{1}{4}\left(\lambda_{040h}-1\right) \\ & - \left(\frac{\lambda_{013h}}{\rho_{xzh}} - 1 \right) \Bigg\} \Bigg] \end{aligned} \quad (8)$$

The $MSE\left(\hat{\sigma}_{R1}^{2}\right)$ to the first order of approximation is, for the optimum value of $d_{h1}(1) = -\frac{d_{h3}^{**}}{d_{h2}^{**}}$ where

$$d_{h2}^{**}=\frac{\lambda_{022h}}{\rho_{xzh}^{2}}+\frac{1}{4}\left(\lambda_{040h}+\lambda_{004h}+2\lambda_{022h}\right)-\frac{\lambda_{031h}}{\rho_{xzh}}-\frac{\lambda_{013h}}{\rho_{xzh}}$$

$$d_{h3}^{**}=\frac{\lambda_{211h}}{\rho_{xzh}}-\frac{1}{2}\left(\lambda_{220h}+\lambda_{202h}\right)+\frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\left(\frac{\lambda_{031h}}{\rho_{xzh}}-\frac{1}{2}\left(\lambda_{040h}+\lambda_{022h}\right)\right)$$

$$MSE_{\min}\left(\hat{\sigma}_{R1}^{2}\right)\cong\sum_{h=1}^{L}\frac{W_h^4}{n_h^3}S_{yh}^4\left[\left\{\left(\lambda_{400h}-1\right)-\frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)}-\frac{\left(d_{h3}^{**}\right)^2}{d_{h2}^{**}}\right\}\right] \quad (9)$$

Estimator of σ_{pst}^2, when S_{xh}^2 and C_{zh} are known,

$$\hat{\sigma}_{S1}^{2}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}s_{yh,lr}^2 f_h\left(k_h\right) \quad (10)$$

where $k_h=\frac{c_{zh}}{C_{zh}}$, and $f_h\left(.\right)$ is the function of k_h such that $f_h\left(1\right)=1$

The $Bias\left(\hat{\sigma}_{S1}^{2}\right)$ to the first order of approximation is given as,

$$\begin{aligned}Bias\left(\hat{\sigma}_{S1}^{2}\right)\cong\sum_{h=1}^{L}\frac{W_h^2}{n^2_h}S_{yh}^2&\Bigg[f_{h1}\left(1\right)\left\{C_{zh}^{\ 2}-\lambda_{201h}C_{zh}+\frac{1}{2}\left(\lambda_{202h}-1\right)-\frac{1}{2}\lambda_{003h}C_{zh}\right.\\&\left.-\frac{1}{8}\left(\lambda_{004h}-1\right)\right\}+f_{h1}\left(1\right)\frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\left\{\lambda_{021h}C_{zh}-\frac{1}{2}\left(\lambda_{022h}-1\right)\right\}\\&+f_{h2}\left(1\right)\left\{\frac{1}{2}C_{zh}^{\ 2}-\frac{1}{2}\lambda_{003h}C_{zh}+\frac{1}{8}\left(\lambda_{004h}-1\right)\right\}\Bigg]\end{aligned} \quad (11)$$

The $MSE\left(\hat{\sigma}_{S1}^{2}\right)$ to the first order of approximation is, for the optimum value of $f_{h1}\left(1\right)=-\frac{f_{h3}^{**}}{f_{h2}^{**}}$, given , where,

$$f_{h2}^{**}=\left(\lambda_{004h}-1\right)+4C_{zh}^2-4C_{zh}\lambda_{003h}\ f_{h3}^{**}=\left(\lambda_{202h}-1\right)-2C_{zh}\lambda_{201h}-\frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\left(\lambda_{022h}-1-2\lambda_{021h}C_{zh}\right)$$

The $MSE\left(\hat{\sigma}_{S1}^{2}\right)_{\min}$ is given by,

$$MSE\left(\hat{\sigma}_{S1}^{2}\right)_{\min}\cong\sum_{h=1}^{L}\frac{W_h^4}{n^3_h}S_{yh}^4\left[\left(\lambda_{400h}-1\right)-\frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)}-\frac{\left(f_{h3}^{**}\right)^2}{f_{h2}^{**}}\right] \quad (12)$$

Estimator of σ^2_{pst}, when S^2_{xh}, $\bar{Z}_h$, S^2_{zh} and ρ_{xzh} are known,

$$\hat{\sigma}^2_{T1} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} s^2_{yh,lr} g_h\left(u_h, \mathrm{v}_h, \tau_h\right) \tag{13}$$

Where $\sigma^2_{yh,lr} = \hat{\sigma}^2_{yh} + \hat{\beta}_{220h}\left(\sigma^2_{xh} - \hat{\sigma}^2_{xh}\right)$, $u_h = \frac{\bar{z}_h}{\bar{Z}_h}$, $v_h = \frac{s^2_{zh}}{S^2_{zh}}$, $\tau_h = \frac{r_{xzh}}{\rho_{xzh}}$ and $g_h\left(u_h, \mathrm{v}_h, \tau_h\right)$ is the function of $\left(u_h, \mathrm{v}_h, \tau_h\right)$ such that $g_h\left(1,1,1\right) = 1$.

The $Bias\left(\hat{\sigma}^2_{T1}\right)$ to the first order of approximation is given as,

$$\begin{aligned} Bias\left(\hat{\sigma}^2_{T1}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n^2_h} S^2_{yh} \Bigg[& g_{h1}\left(1,1,1\right) C_{zh}\left(\lambda_{201h} - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\lambda_{021h}\right) + g_{h2}\left(1,1,1\right) \\ & \times\left(\left(\lambda_{202h}-1\right) - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\left(\lambda_{022h}-1\right)\right) + g_{h3}\left(1,1,1\right)\left(\frac{3}{8}\left(\lambda_{040h}+\lambda_{004h}\right)\right. \\ & \left. - \frac{1}{2}\left(\lambda_{220h}-\lambda_{202h}\right) + \frac{1}{4}\lambda_{022h} + \frac{\lambda_{211h}}{\rho_{xzh}} - \frac{1}{2}\left(\frac{\lambda_{031h}}{\rho_{xzh}} + \frac{\lambda_{013h}}{\rho_{xzh}}\right)\right) + \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} \\ & \times\left(\frac{1}{2}\left(\lambda_{040h}+\lambda_{022h}\right) - \frac{\lambda_{031h}}{\rho_{xzh}}\right) + g_{h11}\left(1,1,1\right)\frac{1}{2}C_{zh}^2 + g_{h22}\left(1,1,1\right)\frac{1}{2}\left(\lambda_{004h}-1\right) \\ & + g_{h33}\left(1,1,1\right)\left(\frac{1}{8}\left(\lambda_{040h}+\lambda_{004h}\right) + \frac{1}{2}\frac{\lambda_{022h}}{\rho^2_{xzh}} + \frac{1}{4}\lambda_{022h} - \frac{1}{2}\left(\frac{\lambda_{031h}}{\rho_{xzh}} + \frac{\lambda_{013h}}{\rho_{xzh}}\right)\right) \\ & + g_{h1,2}\left(1,1,1\right) C_{zh}\lambda_{003h} + g_{h1,3}\left(1,1,1\right)\left(C_{zh}\left(\frac{\lambda_{012h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{021h}+\lambda_{003h}\right)\right)\right) \\ & + g_{h2,3}\left(1,1,1\right)\left(\frac{\lambda_{013h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{004h}+\lambda_{022h}\right)\right)\Bigg] \end{aligned} \tag{14}$$

The $MSE\left(\hat{\sigma}^2_{T1}\right)$ to the first order of approximation is given , where

$$g^{**}_{h0} = \left(\lambda_{400h}-1\right) - \frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)}, \qquad g^{**}_{h3} = \frac{1}{4}\left(\lambda_{040h}+\lambda_{004h}-2\lambda_{022h}\right) + \frac{\lambda_{022h}}{\rho^2_{xzh}} - \frac{1}{\rho_{xzh}}\left(\lambda_{031h}+\lambda_{013h}\right),$$

$$g^{**}_{h4} = \frac{\lambda_{012h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{021h}+\lambda_{003h}\right), \quad g^{**}_{h5} = \frac{\lambda_{013h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{022h}+\lambda_{004h}\right),$$

$$g_{h6}^{**}=\left(\lambda_{201h}-\frac{(\lambda_{220h}-1)}{(\lambda_{040h}-1)}\lambda_{021h}\right),\qquad g_{h7}^{**}=\left((\lambda_{202h}-1)-\frac{(\lambda_{220h}-1)}{(\lambda_{040h}-1)}(\lambda_{022h}-1)\right),$$

$$g_{h8}^{**}=\frac{\lambda_{211h}}{\rho_{xzh}}-\frac{1}{2}(\lambda_{220h}+\lambda_{202h})-\frac{(\lambda_{220h}-1)}{(\lambda_{040h}-1)}\left(\frac{\lambda_{031h}}{\rho_{xzh}}-\frac{1}{2}(\lambda_{040h}+\lambda_{022h})\right).$$

For the optimum values of

$g_{h1}(1,1,1)$

$$=-\frac{(\lambda_{004h}-1)\left(g_{h4}^{**}g_{h8}^{**}-g_{h3}^{**}g_{h6}^{**}\right)+\lambda_{003h}\left(g_{h3}^{**}g_{h7}^{**}-g_{h5}^{**}g_{h8}^{**}\right)+g_{h5}^{**2}g_{h6}^{**}-g_{h4}^{**}g_{h5}^{**}g_{h7}^{**}}{C_{zh}\left\{g_{h3}^{**}\left(1+\lambda_{003h}^{2}-\lambda_{004h}\right)+g_{h4}^{**2}(\lambda_{004h}-1)+g_{h5}^{**2}-2\lambda_{003h}g_{h4}^{**}g_{h5}^{**}\right\}}$$

$g_{h2}(1,1,1)$

$$=-\frac{g_{h4}^{**2}g_{h7}^{**}+g_{h5}^{**}g_{h8}^{**}+\lambda_{003h}\left(g_{h3}^{**}g_{h6}^{**}-g_{h4}^{**}g_{h8}^{**}\right)-g_{h4}^{**}g_{h5}^{**}g_{h6}^{**}-g_{h3}^{**}g_{h7}^{**}}{\left\{g_{h3}^{**}\left(1+\lambda_{003h}^{2}-\lambda_{004h}\right)+g_{h4}^{**2}(\lambda_{004h}-1)+g_{h5}^{**2}-2\lambda_{003h}g_{h4}^{**}g_{h5}^{**}\right\}}$$

$g_{h3}(1,1,1)$

$$=-\frac{g_{h8}^{**}\left(1+\lambda_{003h}^{2}-\lambda_{004h}\right)+g_{h4}^{**}g_{h6}^{**}A_{zh}+g_{h5}^{**}g_{h7}^{**}-\lambda_{003h}\left(g_{h5}^{**}g_{h6}^{**}-g_{h4}^{**}g_{h7}^{**}\right)}{\left\{g_{h3}^{**}\left(1+\lambda_{003h}^{2}-\lambda_{004h}\right)+g_{h4}^{**2}(\lambda_{004h}-1)+g_{h5}^{**2}-2\lambda_{003h}g_{h4}^{**}g_{h5}^{**}\right\}}$$

$$MSE_{\min}\left(\hat{\sigma}_{T1}^{2}\right)\cong\sum_{h=1}^{L}\frac{W_h^4}{n_h^3}S_{yh}^4\left[(\lambda_{400h}-1)-\frac{(\lambda_{220h}-1)^2}{(\lambda_{040h}-1)}-\Omega_h\right]\qquad(15)$$

Where,

$$\Omega_h=\frac{g_{h8}^{**2}\left(1+\lambda_{003h}^{2}-\lambda_{004h}\right)+g_{h6}^{**}(\lambda_{004h}-1)\left(2g_{h4}^{**}g_{h8}^{**}-g_{h3}^{**}g_{h6}^{**}\right)+2\lambda_{003h}\gamma_h+\Delta_h}{g_{h3}^{**}\left(1+\lambda_{003h}^{2}-\lambda_{004h}\right)+g_{h4}^{**2}(\lambda_{004h}-1)+g_{h5}^{**2}-2\lambda_{003h}g_{h4}^{**}g_{h5}^{**}}$$

$$\gamma_h=g_{h3}^{**}g_{h6}^{**}g_{h7}^{**}-g_{h4}^{**}g_{h8}^{**}g_{h7}^{**}-g_{h5}^{**}g_{h6}^{**}g_{h8}^{**}$$

$$\Delta_h=g_{h7}^{**2}\left(g_{h4}^{**2}-g_{h3}^{**}\right)+g_{h5}^{**}\left\{g_{h5}^{**}g_{h6}^{**2}-2g_{h7}^{**}\left(g_{h4}^{**}g_{h6}^{**}+g_{h8}^{**}\right)\right\}$$

In Section 3, we disscused the conditions when measurement error is present in $\left(\sigma_{xh}^2,\overline{Z}_h^*\right)$, $\left(\sigma_{xh}^2,\sigma_{zh}^{2*}\right)$, $\left(\sigma_{xh}^2,\rho_{xzh}\right)$, $\left(\sigma_{xh}^2,C_{zh}^*\right)$ and $\left(\sigma_{xh}^2,\overline{Z}_h^*,S_{zh}^{2*},\rho_{xzh}\right)$ are known. In Sections 4 and 5, efficiency comparison and results are given. In Section 6, we have concluded the results.

PROPOSED FAMILIES OF ESTIMATORS

We discussed following families of estimators as

Estimator of σ_{pst}^{2*}, when σ_{xh}^{2*} and $\overline{Z}_h^*$ are known

In this study, we introduce a new class of estimators for the population variance under post-stratified sampling, leveraging auxiliary information in the presence of measurement error. By incorporating known population characteristics of the auxiliary variable such as its coefficient of variation (CV), coefficient of kurtosis, and mean we derive bias and mean squared error (MSE) expressions and demonstrate the superiority of the proposed estimators over conventional approaches. We propose a family of estimators for post-stratified population variance σ_{pst}^{2*} using information of auxiliary variables under the effect of measurement error, when error is present in Z_h^* by using known σ_{xh}^2 and $\overline{Z}_h^*$ as,

$$\hat{\sigma}_{P1}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \hat{\sigma}_{yh,lr}^2 a_h\left(u_h^*\right) \tag{16}$$

Where $\hat{\sigma}_{yh,lr}^2 = \hat{\sigma}_{yh}^2 + \hat{\beta}_{220h}\left(\sigma_{xh}^2 - \hat{\sigma}_{xh}^2\right)$, $u_h^* = \dfrac{\overline{z}_h^*}{\overline{Z}_h^*}$ and also $a_h(.)$ is the function of u_h^* that is $a_h(1) = 1$. It also satisfies the regularity conditions as mentioned by Srivastava (1971), Singh and Karpe (2008) and Masood and Shabbir (2015).

$$\hat{\sigma}_{P1}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h}\left[\hat{\sigma}_{yh}^2 + \hat{\beta}_{220h}\left(\sigma_{xh}^2 - \hat{\sigma}_{xh}^2\right)\right] a_h\left(\frac{\overline{z}_h^*}{\overline{Z}_h^*}\right) \tag{17}$$

The $Bias\left(\hat{\sigma}_{P1}^{2*}\right)$ to first order of approximation, is given by

$$Bias\left(\hat{\sigma}_{P1}^{2\,*}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h^{\,2}} \sigma_{yh}^2 \left[a_{h1}(1) C_{zh}\lambda_{102h} - a_{h1}(1)\frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} C_{zh}\lambda_{021h} + a_{h2}(1)\frac{1}{2}A_{zh}^{'}\right] \tag{18}$$

The $MSE\left(\hat{\sigma}_{P1}^{2*}\right)$ to the first order of approximation, is given by

$$MSE\left(\hat{\sigma}_{P1}^{2*}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n_h^{\,3}} \sigma_{yh}^4 \left[\left(\lambda_{400h}-1\right) - \frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)} + A_{zh}^{'} + 2\left\{\lambda_{201h} - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\lambda_{021h}\right\}\right] \tag{19}$$

For the optimum value $a_{h1}(1) = -\frac{1}{A'_{zh}}\left\{C_{zh}\lambda_{201h} - \frac{(\lambda_{220h}-1)}{(\lambda_{040h}-1)}C_{zh}\lambda_{021h}\right\}$, the $MSE\left(\hat{\sigma}^{2*}_{P1}\right)_{\min}$ is given by,

$$MSE\left(\hat{\sigma}^{2*}_{P1}\right)_{\min} \cong \sum_{h=1}^{L}\frac{W_h^4}{n_h^3}\sigma^4_{yh}\left[(\lambda_{400h}-1) - \frac{(\lambda_{220h}-1)^2}{(\lambda_{040h}-1)} - \frac{1}{A'_{zh}}\left\{\lambda_{201h} - \frac{(\lambda_{220h}-1)}{(\lambda_{040h}-1)}\lambda_{021h}\right\}^2\right] \quad (20)$$

The following estimators also belong to above class of estimators which gives the same results as

$$\hat{\sigma}^{2*}_{P2} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[1+a_h\left(u_h^*-1\right)\right]^{-1}, \qquad \hat{\sigma}^{2*}_{P3} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[a_h+(1-a_h)u_h^{*-1}\right],$$

$$\hat{\sigma}^{2*}_{P4} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[(1-a_h)u_h^{2*}+a_hu_h^*\right]^{-1}, \qquad \hat{\sigma}^{2*}_{P5} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[(1+a_h)-a_hu_h^*\right],$$

$$\hat{\sigma}^{2*}_{P6} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{u_h^*}{1+(1+a_h)\left(u_h^*-1\right)}\right], \qquad \hat{\sigma}^{2*}_{P7} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[a_hu_h^{-1*}+(1-a_h)u_h^*\right],$$

$$\hat{\sigma}^{2*}_{P8} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{u_h^*}{(1-a_h)u_h^*+a_h}\right], \qquad \hat{\sigma}^{2*}_{P9} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[2-u_h^{a_h*}\right],$$

$$\hat{\sigma}^{2*}_{P10} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[a_h+(1-a_h)u_h^*\right], \qquad \hat{\sigma}^{2*}_{P11} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{u_h^*(1-a_h)+a_h}{(1-a_h)+a_hu_h^*}\right],$$

$$\hat{\sigma}^{2*}_{P12} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{1+\pi_1\left(u_h^*-1\right)}{1+\pi_2\left(u_h^*-1\right)}\right]^{a_h}, \qquad \hat{\sigma}^{2*}_{P13} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[u_h^{a_h*}\right].$$

(where π_1 and π_2 are constants)

Estimator of σ^{2*}_{pst} , σ^2_{xh} and σ^{2*}_{zh} are known

We proposed a family of estimators for post-stratified population variance σ^{2*}_{pst} under the effect of measurement error when error is present in Z_h^* and σ^{2*}_{zh} by using known σ^2_{xh} and σ^{2*}_{zh} as,

$$\hat{\sigma}^{2*}_{Q1} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\hat{\sigma}^2_{yh,lr}b_h\left(v_h^*\right) \quad (21)$$

where $v_h^* = \dfrac{\sigma_{zh}^{2*}}{\sigma_{zh}^2}$ and $b_h(.)$ is the function of v_h^* such that $b_h(1) = 1$. It satisfies all the regularity conditions mentioned by Srivastava (1971), Singh and Karpe (2008) and Masood and Shabbir (2015).

$$\hat{\sigma}_{Q1}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \left[\hat{\sigma}_{yh}^2 + \hat{\beta}_{220h} \left(\sigma_{xh}^2 - \hat{\sigma}_{xh}^2 \right) \right] b_h \left(\frac{\hat{\sigma}_{zh}^{2*}}{\sigma_{zh}^{2*}} \right) \tag{22}$$

The $Bias\left(\hat{\sigma}_{Q1}^{2*}\right)$ to first order of approximation, is given by

$$Bias\left(\hat{\sigma}_{Q1}^{2*}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h^2} \sigma_{yh}^2 \left[b_{h1}(1) \left\{ \left(\lambda_{202h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)}{\left(\lambda_{040h} - 1\right)} \left(\lambda_{022h} - 1\right) \right\} + b_{h\,2}(1) \frac{1}{2} A_{zh} \right] \tag{23}$$

The $MSE\left(\hat{\sigma}_{Q1}^{2*}\right)$ to the first order of approximation, is given by

$$MSE\left(\hat{\sigma}_{Q1}^{2*}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} \sigma_{yh}^4 \left[\left(\lambda_{400h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)^2}{\left(\lambda_{040h} - 1\right)} + b_{h1}(1)^2 A_{zh} \right.$$
$$\left. +2\left(b_{h1}(1) \left(\lambda_{202h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)}{\left(\lambda_{040h} - 1\right)} \left(\lambda_{022h} - 1\right) \right) \right] \tag{24}$$

For the optimum value of $b_{h1}(1) = -\dfrac{1}{A_{zh}} \left\{ \left(\lambda_{202h} - 1\right) - \dfrac{\left(\lambda_{220h} - 1\right)}{\left(\lambda_{040h} - 1\right)} \left(\lambda_{022h} - 1\right) \right\}$

The $MSE\left(\hat{\sigma}_{Q1}^{2*}\right)_{\min}$ is given by,

$$MSE_{\min}\left(\hat{\sigma}_{Q1}^{2*}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} \sigma_{yh}^4 \left[\left(\lambda_{400h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)^2}{\left(\lambda_{040h} - 1\right)} - \frac{1}{A_{zh}} \right.$$
$$\left. \left\{ \left(\lambda_{202h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)}{\left(\lambda_{040h} - 1\right)} \left(\lambda_{022h} - 1\right) \right\}^2 \right] \tag{25}$$

The following estimators belong to above class of estimators which give the same results as

$$\hat{\sigma}_{Q2}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{yh,lr}^{2*} \left[1 + b_h \left(v_h^* - 1 \right) \right]^{-1}, \qquad \hat{\sigma}_{Q3}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{yh,lr}^{2*} \left[b_h + \left(1 - b_h\right) v_h^{*-1} \right],$$

$$\hat{\sigma}_{Q4}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{yh,lr}^{2*} \left[\left(1 - b_h\right) v_h^{*2} + b_h v_h^* \right]^{-1}, \qquad \hat{\sigma}_{Q5}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{yh,lr}^{2*} \left[\left(1 + b_h\right) - b_h v_h^* \right],$$

$$\hat{\sigma}^{2*}_{Q6}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{v_h^*}{1+\left(1+b_h\right)\left(v_h^*-1\right)}\right], \quad \hat{\sigma}^{2*}_{Q7}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[b_h v_h^{-1*}+(1-b_h)v_h^*\right],$$

$$\hat{\sigma}^{2*}_{Q8}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{v_h^*}{\left(1-b_h\right)v_h^*+b_h}\right], \quad \hat{\sigma}^{2*}_{Q9}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[2-v_h^{*b_h}\right]$$

$$\hat{\sigma}^{2*}_{Q10}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[b_h+\left(1-b_h\right)v_h^*\right], \quad \hat{\sigma}^{2*}_{Q11}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{v_h^*\left(1-b_h\right)+b_h}{\left(1-b_h\right)+b_h v_h^*}\right],$$

$$\hat{\sigma}^{2*}_{Q12}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[\frac{1+\pi_1\left(v_h^*-1\right)}{1+\pi_2\left(v_h^*-1\right)}\right]^{b_h}, \quad \hat{\sigma}^{2*}_{Q13}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^{2*}_{yh,lr}\left[v_h^{*b_h}\right].$$

(where π_1 and π_2 are constants)

Estimator of σ^{2*}_{pst}, when σ^2_{xh} and ρ^*_{xzh} are known.

We proposed a family of estimators for post-stratified population variance σ^{2*}_{pst} under the effect of measurement error when error is present in Z_h^* by using known σ^2_{xh} and ρ^*_{xzh} as

$$\hat{\sigma}^{2*}_{R1}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma^2_{yh,lr}d_h\left(\tau_h^*\right) \tag{26}$$

Where $\tau_h^*=\frac{r_{xzh}}{\rho_{xzh}}$ and $d_h\left(1\right)$ is the function of τ_h^* such that $d_h\left(1\right)=1$. It satisfies all the regularity conditions mentioned by Srivastava (1971), Singh and Karpe (2008) and Masood and Shabbir (2015).

$$\hat{\sigma}^{2*}_{R1}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\left[\hat{\sigma}^2_{yh}+\hat{\beta}_{220h}\left(\sigma^2_{xh}-\hat{\sigma}^2_{xh}\right)\right]d_h\left(\frac{r^*_{xzh}}{\rho^*_{xzh}}\right) \tag{27}$$

The $Bias\left(\hat{\sigma}^{2*}_{R1}\right)$ to first order of approximation, is given by

$$Bias\left(\hat{\sigma}^{2*}_{R1}\right)\cong\sum_{h=1}^{L}\frac{W_h^2}{n_h^2}\sigma^2_{yh}\left[\left\{d_h\left(1\right)\left(-\frac{\left(\lambda_{220h}-1\right)}{2}+\frac{3}{8}\left(\lambda_{040h}-1\right)-\frac{1}{2}\left(\lambda_{202h}-1\right)\right.\right.\right.$$

$$\left.-\frac{1}{2}\left(\frac{\lambda_{013h}}{\rho_{xzh}}-1\right)+\left(\frac{\lambda_{211h}}{\rho_{xzh}}-1\right)-\frac{1}{2}\left(\frac{\lambda_{031h}}{\rho_{xzh}}-1\right)+\frac{1}{4}\left(\lambda_{022h}-1\right)\right.$$

$$+\frac{3}{8}A_{zh}+\left\{\frac{d_{h2}}{2}\left(\frac{1}{2}\left(\lambda_{022h}-1\right)-\left(\frac{\lambda_{031h}}{\rho_{xzh}}-1\right)+\frac{1}{4}A_{zh}-\left(\frac{\lambda_{013h}}{\rho_{xzh}}-1\right)\right.\right.$$

$$\left.\left.\times\left(\frac{\lambda_{022}}{\rho_{xzh}^{2}}-1\right)+\frac{1}{4}\left(\lambda_{040h}-1\right)\right\}\right] \quad (28)$$

The $MSE\left(\hat{\sigma}_{R1}^{2*}\right)$ to the first order of approximation, is given by

$$MSE\left(\hat{\sigma}_{R1}^{2*}\right)\cong\sum_{h=1}^{L}\frac{W_h^4}{n_h^3}\sigma_{yh}^4\left[\left\{\left(\lambda_{400h}-1\right)-\frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)}+d_{h1}^2\left(1\right)d_{h2}^{**}+2d_{h1}\left(1\right)d_{h3}^{**}\right\}^2\right] \quad (29)$$

Where

$$d_{h2}^{**}=\frac{\lambda_{022h}}{\rho_{xzh}^2}+\frac{1}{4}\left(\lambda_{040h}+A_{zh}+\lambda_{022h}\right)-\frac{\lambda_{031h}}{\rho_{xzh}}-\frac{\lambda_{013h}}{\rho_{xzh}}$$

$$d_{h3}^{**}=\frac{\lambda_{211h}}{\rho_{xzh}}-\frac{1}{2}\left(\lambda_{220h}+\lambda_{202h}\right)+\frac{\left(\lambda_{220h}-1\right)}{2\left(\lambda_{040h}-1\right)}\left(\frac{\lambda_{031h}}{\rho_{xzh}}-\frac{1}{2}\left(\lambda_{040h}+\lambda_{022h}\right)\right).$$

Now for the optimum value of $d_h\left(1\right)=-\frac{d_{h3}^{**}}{d_{h2}^{**}}$

The $MSE_{\min}\left(\hat{\sigma}_{R1}^{2*}\right)$ is given by,

$$MSE_{\min}\left(\hat{\sigma}_{R1}^{2*}\right)\cong\sum_{h=1}^{L}\frac{W_h^4}{n_h^3}\sigma_{yh}^4\left[\left\{\left(\lambda_{400h}-1\right)-\frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)}-\frac{\left(d_{h3}^{**}\right)^2}{d_{h2}^{**}}\right\}\right] \quad (30)$$

The following estimators belong to above class of estimators which give the same results as

$$\hat{\sigma}_{R2}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[1+d_h\left(\tau_h^*-1\right)\right]^{-1}, \qquad \hat{\sigma}_{R3}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[d_h+\left(1-d_h\right)\tau_h^{*-1}\right],$$

$$\hat{\sigma}_{R4}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\left(1-d_h\right)\tau_h^{*2}+d_h\tau_h^*\right]^{-1}, \qquad \hat{\sigma}_{R5}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\left(1+d_h\right)-d_h\tau_h^*\right],$$

$$\hat{\sigma}_{R6}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\frac{\tau_h^*}{1+\left(1+d_h\right)\left(\tau_h^*-1\right)}\right], \qquad \hat{\sigma}_{R7}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[d_h\tau_h^{*-1}+(1-d_h)\tau_h^*\right],$$

$$\hat{\sigma}_{R8}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\frac{\tau_h^*}{\left(1-d_h\right)\tau_h^*+d_h}\right], \qquad \hat{\sigma}_{R9}^{2*}=\sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[2-\tau_h^{*d_h}\right],$$

$$\hat{\sigma}^{2*}_{R10} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma^{2*}_{yh,lr} \left[d_h + \left(1-d_h\right)\tau_h^* \right], \qquad \hat{\sigma}^{2*}_{R11} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma^{2*}_{yh,lr} \left[\frac{\tau_h^*\left(1-d_h\right)+d_h}{\left(1-d_h\right)+d_h\tau_h^*} \right],$$

$$\hat{\sigma}^{2*}_{R12} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma^{2*}_{yh,lr} \left[\frac{1+\pi_1\left(\tau_h^*-1\right)}{1+\pi_2\left(\tau_h^*-1\right)} \right]^{d_h}, \qquad \hat{\sigma}^{2*}_{R13} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma^{2*}_{yh,lr} \left[\tau_h^{*d_h} \right].$$

(where π_1 and π_2 are constants)

Estimator of σ^{2*}_{pst}, when σ^2_{xh} and C^*_{zh} are known

We proposed a family of estimators for post-stratified population variance σ^{2*}_{pst} under the effect of measurement error, when error present in Z_h^* and C^*_{zh} by using known σ^2_{xh} and C^*_{zh} as:

$$\hat{\sigma}^{2*}_{S1} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma^2_{yh,lr} f_h\left(k_h^*\right) \qquad (31)$$

Where $k_h = \frac{c^*_{zh}}{C^*_{zh}}$ and $f_h(.)$ is the function of k_h^* such that $f_h(1)=1$. It also satisfies the regularity conditions as mentioned by Srivastava (1971), Singh and Karpe (2008) and Masood and Shabir (2015).

$$\hat{\sigma}^{2*}_{S1} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \left[\hat{\sigma}^2_{yh} + \hat{\beta}_{220h}\left(\sigma^2_{xh} - \hat{\sigma}^2_{xh}\right) \right] f_h\left(\frac{c^*_{zh}}{C^*_{zh}}\right) \qquad (32)$$

The $Bias\left(\hat{\sigma}^{2*}_{S1}\right)$ to the first order of approximation, is given by,

$$Bias\left(\hat{\sigma}^{2*}_{S1}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma^2_{yh} \left[f_h(1)\left\{ \left(-C_{zh}\lambda_{201h} - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} C_{zh}\lambda_{021} \right) + A'_{zh} \right.\right.$$
$$\left. + \frac{1}{2}\left(\lambda_{220h}-1\right) - \frac{1}{2}\left(\lambda_{022h}-1\right)\frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)} - \frac{1}{4}A_{zh} \right\}$$
$$\left. + \frac{f_h(1)^2}{2}\left\{ A'_{zh} + \frac{1}{4}A_{zh} - \lambda_{003h}C_{zh} \right\} \right] \qquad (33)$$

The $MSE\left(\hat{\sigma}^{2*}_{S1}\right)$ to the first order of approximation, is given by

$$MSE\left(\hat{\sigma}^{2*}_{S1}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n^3_h} \sigma^4_{yh} \left[f_h(1)\left\{ \left(\lambda_{400h}-1\right) - \frac{\left(\lambda_{220h}-1\right)^2}{\left(\lambda_{040h}-1\right)} + f^{**}_{h2} + 2f^{**}_{h3} \right\} \right] \qquad (34)$$

where,

$$f_{h2}^{**} = A_{zh} + 4A_{zh}^{'} - 4C_{zh}\lambda_{003h} \quad f_{h3}^{**} = (\lambda_{202h} - 1) - 2C_{zh}\lambda_{201h} - \frac{(\lambda_{220h} - 1)}{(\lambda_{040h} - 1)}\{(\lambda_{022h} - 1) - 2\lambda_{021h}C_{zh}\}$$

Now for the optimum value of $f_h(1) = -\frac{f_{h3}^{**}}{f_{h2}^{**}}$,

The $MSE\left(\hat{\sigma}_{S1}^{2\,*}\right)_{\min}$ is given by,

$$MSE_{\min}\left(\hat{\sigma}_{S1}^{2*}\right) = \sum_{h=1}^{L}\frac{W_h^4}{n_h^3}\sigma_{yh}^4\left[(\lambda_{400h} - 1) - \frac{(\lambda_{220h} - 1)^2}{(\lambda_{040h} - 1)} - \frac{\left(f_{h3}^{**}\right)^2}{f_{h2}^{**}}\right] \tag{35}$$

The following estimators belong to above class of estimators which give the same results as

$$\hat{\sigma}_{S2}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[1 + f_h\left(k_h^* - 1\right)\right]^{-1} \qquad \hat{\sigma}_{S3}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[f_h + (1 - f_h)k_h^{*-1}\right]$$

$$\hat{\sigma}_{S4}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[(1 - f_h)k_h^{2*} + f_h k_h^*\right]^{-1}, \qquad \hat{\sigma}_{S5}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[(1 + f_h) - f_h k_h^*\right],$$

$$\hat{\sigma}_{S6}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\frac{k_h^*}{1 + (1 + f_h)\left(k_h^* - 1\right)}\right], \qquad \hat{\sigma}_{S7}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[f_h k_h^{*-1} + (1 - f_h)k_h^*\right],$$

$$\hat{\sigma}_{S8}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\frac{k_h^*}{(1 - f_h)k_h^* + f_h}\right], \qquad \hat{\sigma}_{S9}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[2 - k_h^{*f_h}\right],$$

$$\hat{\sigma}_{S10}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[f_h + (1 - f_h)k_h^*\right], \qquad \hat{\sigma}_{S11}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\frac{k_h^*(1 - f_h) + f_h}{(1 - f_h) + f_h k_h^*}\right],$$

$$\hat{\sigma}_{S12}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[\frac{1 + \pi_1\left(k_h^* - 1\right)}{1 + \pi_2\left(k_h^* - 1\right)}\right]^{f_h}, \qquad \hat{\sigma}_{S13}^{2*} = \sum_{h=1}^{L}\frac{W_h^2}{n_h}\sigma_{yh,lr}^{2*}\left[k_h^{*f_h}\right].$$

(where π_1 and π_2 are constants)

Estimator of $\hat{\sigma}_{pst}^{2*}$, when $\overline{Z}_h^*$, σ_{zh}^{2*} and ρ_{xzh}^* are known

We proposed a family of estimators for post-stratified population variance σ_{Pst}^{2*} under the effect of measurement error, when error is present in Z_h^* and σ_{zh}^{2*} by using known σ_{xh}^2, $\overline{Z}_h^*$, σ_{zh}^{2*} and ρ_{xzh}^* as

$$\hat{\sigma}_{T1}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{yh,lr}^2 g_h\left(u_h^*, \mathrm{v}_h^*, \tau_h^*\right) \tag{36}$$

Where $\sigma_{yh,lr}^2 = \hat{\sigma}_{yh}^2 + \hat{\beta}_{220h}\left(\sigma_{xh}^2 - \hat{\sigma}_{xh}^2\right)$, $\tau_h = \frac{r_{xzh}^*}{\rho_{xzh}^*}$ and $g_h\left(u_h^*, \mathrm{v}_h^*, \tau_h^*\right)$ is the function of $\left(u_h^*, \mathrm{v}_h^*, \tau_h^*\right)$ such that $g_h\left(1,1,1\right) = 1$. It also satisfies the regularity conditions as mentioned by Srivastava (1971), Singh and Karpe (2008) and Masood and Shabbir (2015).

$$\hat{\sigma}_{T1}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h}\left[\hat{\sigma}_{yh}^2 + \hat{\beta}_{220h}\left(\sigma_{xh}^2 - \hat{\sigma}_{xh}^2\right)\right] g_h\left(\frac{\bar{z}_h^*}{\bar{Z}_h^*}, \frac{\sigma_{zh}^{2*}}{\sigma_{zh}^{2*}}, \frac{r_{xzh}^*}{\rho_{xzh}^*}\right) \tag{37}$$

The $Bias\left(\hat{\sigma}_{T1}^{2*}\right)$ to the first order of approximation, is given by,

$$\begin{aligned}
Bias\left(\hat{\sigma}_{T1}^{2*}\right) \cong \sum_{h=1}^{L} \frac{W_h^2}{n_h^2}\Bigg[& g_{h1}\left(1,1,1\right)\left(C_{zh}\left(\lambda_{201h} - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\lambda_{021h}\right)\right. \\
& + g_{h2}\left(1,1,1\right)\left(\left(\lambda_{202h}-1\right) - \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\left(\lambda_{022h}-1\right)\right) \\
& + g_{h3}\left(1,1,1\right)\left(\frac{3}{8}\left(\lambda_{040h} - A_{zh}\right) - \frac{1}{2}\left(\lambda_{220h} + \lambda_{202h}\right) + \frac{1}{4}\lambda_{012h}\right. \\
& + \frac{\lambda_{211h}}{\rho_{xzh}} - \frac{1}{2}\left(\frac{\lambda_{031h}}{\rho_{xzh}} + \frac{\lambda_{013h}}{\rho_{xzh}}\right) + \frac{\left(\lambda_{220h}-1\right)}{\left(\lambda_{040h}-1\right)}\left(\frac{1}{2}\left(\lambda_{040h} + \lambda_{022h}\right)\right. \\
& \left.\left. - \frac{\lambda_{031h}}{\rho_{xzh}}\right)\right) + g_{h11}\left(1,1,1\right)\frac{A_{zh}'}{2} + g_{h22}\left(1,1,1\right)\frac{A_{zh}}{2} \\
& + g_{h33}\left(1,1,1\right)\left(\frac{1}{8}\left(\lambda_{040h} + A_{zh}\right) + \frac{1}{4}\lambda_{022h} + \frac{1}{2}\frac{\lambda_{022h}}{\rho_{xzh}^2}\right. \\
& \left. - \frac{1}{2}\left(\frac{\lambda_{031h}}{\rho_{xzh}} + \frac{\lambda_{013h}}{\rho_{xzh}}\right)\right) + g_{h12}\left(1,1,1\right)C_{zh}\lambda_{003h} \\
& + g_{h13}\left(1,1,1\right)C_{zh}\left(\frac{\lambda_{012h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{021h} + \lambda_{003h}\right)\right) \\
& + g_{h23}\left(1,1,1\right)\left(\frac{\lambda_{013h}}{\rho_{xzh}} - \frac{1}{2}\left(A_{zh} + \lambda_{022h}\right)\right)
\end{aligned} \tag{38}$$

The $MSE\left(\hat{\sigma}_{T1}^{2*}\right)$ to the first order of approximation, is given by

$$MSE\left(\hat{\sigma}_{T1}^{2*}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} \sigma_{yh}^4 \left[g_{h0}^{**} + C_{zh}^2\, g_{h1}(1,1,1)^2 + A_{zh} g_{h2}(1,1,1)^2 + g_{h3}(1,1,1)^2\, g_{h3}^{**} \right.$$
$$+ 2g_{h1}(1,1,1)\, C_{zh} g_{h6}^{**} + 2g_{h2}(1,1,1)\, g_{h7}^{**} + 2g_{h3}(1,1,1)\, g_{h8}^{**}$$
$$+ 2g_{h1}(1,1,1)\, g_{h2}(1,1,1)\, C_{zh}\lambda_{003h} + 2g_{h1}(1,1,1)\, g_{h3}(1,1,1)\, C_{zh} g_{h4}^{**}$$
$$\left. + 2g_{h2}(1,1,1)\, g_{h3}(1,1,1)\, g_{h5}^{**} \right] \qquad (39)$$

Where,

$$g_{h0}^{**} = \left(\lambda_{400h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)^2}{\left(\lambda_{040h} - 1\right)}, \qquad g_{h3}^{**} = \frac{1}{4}\left(\lambda_{040h} + A_{zh} - 2\lambda_{022h}\right) + \frac{\lambda_{022h}}{\rho_{xzh}^2} - \frac{1}{\rho_{xzh}}\left(\lambda_{031h} + \lambda_{013h}\right),$$

$$g_{h4}^{**} = \frac{\lambda_{012h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{021h} + \lambda_{003h}\right), \; g_{h5}^{**} = \frac{\lambda_{013h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{022h} + A_{zh}\right),$$

$$g_{h6}^{**} = \left(\lambda_{201h} - \frac{\left(\lambda_{220h} - 1\right)}{\left(\lambda_{040h} - 1\right)}\lambda_{021h}\right), \; g_{h7}^{**} = \left(\left(\lambda_{040h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)}{\left(\lambda_{040h} - 1\right)}\left(\lambda_{022h} - 1\right)\right),$$

$$g_{h8}^{**} = \frac{\lambda_{211h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{220h} + \lambda_{202h}\right) - \frac{\left(\lambda_{220h} - 1\right)}{\left(\lambda_{040h} - 1\right)}\left(\frac{\lambda_{031h}}{\rho_{xzh}} - \frac{1}{2}\left(\lambda_{040h} + \lambda_{022h}\right)\right).$$

By elimination method

$$g_{h1}(1,1,1)$$
$$= -\frac{A_{zh}\left(g_{h4}^{**} g_{h8}^{**} - g_{h3}^{**} g_{h6}^{**}\right) + \lambda_{003h}\left(g_{h3}^{**} g_{h7}^{**} - g_{h5}^{**} g_{h8}^{**}\right) + g_{h5}^{**2} g_{h6}^{**} - g_{h4}^{**} g_{h5}^{**} g_{h7}^{**}}{C_{zh}\left\{g_{h3}^{**}\left(1 + \lambda_{003h}^2 - A_{zh}\right) + g_{h4}^{**2} A_{zh} + g_{h5}^{**2} - 2\lambda_{003h} g_{h4}^{**} g_{h5}^{**}\right\}}$$

$$g_{h2}(1,1,1)$$
$$= -\frac{g_{h4}^{**2} g_{h7}^{**} + g_{h5}^{**} g_{h8}^{**} + \lambda_{003h}\left(g_{h3}^{**} g_{h6}^{**} - g_{h4}^{**} g_{h8}^{**}\right) - g_{h4}^{**} g_{h5}^{**} g_{h6}^{**} - g_{h3}^{**} g_{h7}^{**}}{\left\{g_{h3}^{**}\left(1 + \lambda_{003h}^2 - A_{zh}\right) + g_{h4}^{**2} A_{zh} + g_{h5}^{**2} - 2\lambda_{003h} g_{h4}^{**} g_{h5}^{**}\right\}}$$

$$g_{h3}(1,1,1)$$
$$= -\frac{g_{h8}^{**}\left(1 + \lambda_{003h}^2 - A_{zh}\right) + g_{h4}^{**} g_{h6}^{**} A_{zh} + g_{h5}^{**} g_{h7}^{**} - \lambda_{003h}\left(g_{h5}^{**} g_{h6}^{**} - g_{h4}^{**} g_{h7}^{**}\right)}{\left\{g_{h3}^{**}\left(1 + \lambda_{003h}^2 - A_{zh}\right) + g_{h4}^{**2} A_{zh} + g_{h5}^{**2} - 2\lambda_{003h} g_{h4}^{**} g_{h5}^{**}\right\}}$$

Now for the optimum value of the $MSE\left(\hat{\sigma}_{T1}^{2*}\right)_{\min}$ is given by,

$$MSE_{\min}\left(\hat{\sigma}_{\mathrm{T1}}^{2\,*}\right) \cong \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} \sigma_{yh}^4 \left[\left(\lambda_{400h} - 1\right) - \frac{\left(\lambda_{220h} - 1\right)^2}{\left(\lambda_{040h} - 1\right)} - \Omega_h^* \right] \qquad (40)$$

Where,

$$\Omega_h^* = -\frac{g_{h8}^{**2}\left(1+\lambda_{003h}^2 - A_{zh}\right) + g_{h6}^{**} A_{zh}\left(2 g_{h4}^{**} g_{h8}^{**} - g_{h3}^{**} g_{h6}^{**}\right) + 2\lambda_{003h}\gamma_h + \Delta_h}{g_{h3}^{**}\left(1+\lambda_{003h}^2 - A_{zh}\right) + g_{h4}^{**2} A_{zh} + g_{h5}^{**2} - 2\lambda_{003h} g_{h4}^{**} g_{h5}^{**}}$$

$$\gamma_h = g_{h3}^{**} g_{h6}^{**} g_{h7}^{**} - g_{h4}^{**} g_{h8}^{**} g_{h7}^{**} - g_{h5}^{**} g_{h6}^{**} g_{h8}^{**}$$

$$\Delta_h = g_{h7}^{**2}\left(g_{h4}^{**2} - g_{h3}^{**}\right) + g_{h5}^{**}\left\{g_{h5}^{**} g_{h6}^{**2} - 2 g_{h7}^{**}\left(g_{h4}^{**} g_{h6}^{**} + g_{h8}^{**}\right)\right\}$$

The following estimator belong to above class of estimator which give the same result as

$$\hat{\sigma}_{T2}^{2*} = \sum_{h=1}^{L} \frac{W_h^2}{n_h} \sigma_{yh,lr}^{2*} \left(u_h^*\right)^{g_{h1}(1,1,1)} \left(\mathrm{v}_h^*\right)^{g_{h2}(1,1,1)} \left(\tau_h^*\right)^{g_{h3}(1,1,1)} \quad .$$

EFFICIENCY COMPARISON

We compare the *MSE* of proposed estimators with existing estimators as

i. By (3) and (20)

$$MSE_{\min}\left(\hat{\sigma}_{P1}^2\right) - MSE_{\min}\left(\hat{\sigma}_{P1}^{2*}\right)$$

$$= \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} \sigma_{yh}^4 \left[\frac{1}{A_{zh}^{'}} - 1 \right] > 0. \qquad (41)$$

ii. By (6) and (25)

$$MSE\left(\hat{\sigma}_{Q1}^2\right) - MSE\left(\hat{\sigma}_{Q1}^{2*}\right)$$

$$= \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} \sigma_{yh}^4 \left[\frac{1}{A_{zh}} - \frac{1}{\left(\lambda_{400h} - 1\right)} \right] > 0 \qquad (42)$$

iii. By (9) and (30)

$$MSE\left(\hat{\sigma}_{R1}^2\right) - MSE\left(\hat{\sigma}_{R1}^{2*}\right)$$

$$= \sum_{h=1}^{L} \frac{W_h^4}{n_h^3} \sigma_{yh}^4 \left[\frac{\left(d_{h3}^{**}\right)^2}{d_{h2}^{**}} \right] > 0 \qquad (43)$$

iv. By (12) and (35)

$$MSE\left(\hat{\sigma}^2_{S1}\right) - MSE\left(\hat{\sigma}^{2*}_{S1}\right)$$

$$= \sum_{h=1}^{L} \frac{W_h^4}{n^3{}_h} \sigma_{yh}^4 \left[\frac{\left(f_{h3}^{**}\right)^2}{f_{h2}^{**}} \right] > 0 \qquad (44)$$

v. By (15) and (40)

$$MSE\left(\hat{\sigma}^2_{T1}\right) - MSE\left(\hat{\sigma}^{2*}_{T1}\right)$$

$$= \sum_{h=1}^{L} \frac{W_h^4}{n^3{}_h} \sigma_{yh}^4 \left[\Omega_h\right] > 0. \qquad (45)$$

DATA DESCRIPTION

Data Set 1

The empirical analysis is based on a widely used benchmark dataset originally introduced by Murthy (1967), which comprises information on output (y), labor input (x, representing the number of workers), and fixed capital (z) for a sample of 80 factories. The population is divided into four strata based on size and sectoral classification, allowing for post-stratified estimation techniques. While the data originates from a historical source, it has been extensively reused in contemporary methodological studies due to its real-world structure, variability, and suitability for evaluating estimator performance under complex sampling designs (Singh & Karan, 2021; Khan et al., 2022; Riaz et al., 2023).

All the details, see Appendix B.

Table 1 : *MSE* of proposed and existing class of estimators

Estimators	*MSE*	*MSE*(with measurement error)	Absolute difference
$\hat{\sigma}^{2*}_{P1}$	11298865	15957810	4658944
$\hat{\sigma}^{2*}_{Q1}$	10856985	12755722	1898737
$\hat{\sigma}^{2*}_{R1}$	19763629	19740598	23031
$\hat{\sigma}^{2*}_{S1}$	15305362	18675694	3370331
$\hat{\sigma}^{2*}_{T1}$	23317330	24388888	1071559

Data Set 2

Source: Kadilar and Cingi (2006): Let y = the level of apple production (1 unit = 100 tons), x = the number of apple trees in 1999 and z = the number of apple trees in 1998 (1unit = 100 trees) of 106 villages in the Marmarian Region and in 854 villages consisting of 6 strata, respectively (as 1: Marmarian, 2: Agean, 3: Mediterranean, 4: Central Anatolia,5: Black Sea, 6: East and Southeast Anatolia). . For details, see Appendix B.

Table 2: *MSE* of proposed and suggested class of estimators

Estimators	*MSE*(without measurement error)	*MSE*(with measurement error)	Absolute difference
$\hat{\sigma}^{2*}_{P1}$	340844.03×10^6	355583.65×10^6	14739.62×10^6
$\hat{\sigma}^{2*}_{Q1}$	355520.40×10^6	355557.80×10^6	37.3822×10^6
$\hat{\sigma}^{2*}_{R1}$	351518.71×10^6	359786.88×10^6	8268.17×10^6
$\hat{\sigma}^{2*}_{S1}$	303846.3258×10^6	355499.62×10^6	51653.29×10^6
$\hat{\sigma}^{2*}_{T1}$	275264.76×10^6	322492.78×10^6	47228.03×10^6

CONCLUSION

This study investigates the estimation of post-stratified population variance using two auxiliary variables under the realistic condition that one or more of these variables are subject to measurement error—a common yet often overlooked issue in survey practice. Contrary to the traditional assumption of error-free auxiliary data, real-world datasets—especially in economic, social, and industrial surveys—are frequently contaminated by non-sampling errors**,** including respondent misreporting, instrument inaccuracy, data entry mistakes, and proxy reporting (Fuller, 2009; Buonaccorsi, 2023; Lohr, 2022).

Our findings reveal that ignoring measurement error leads to biased and misleading inferences**,** even when using otherwise efficient estimator families. While existing classes of post-stratified variance estimators perform well under idealized conditions (i.e., no measurement error), their performance deteriorates significantly in the presence of data inaccuracies. The proposed family of estimators, although theoretically sound under classical assumptions, exhibits inflated mean squared error (MSE) when applied to error-contaminated auxiliary variables, resulting in overestimation or underestimation of the actual population variance.

As demonstrated in Tables 1 and 2, the magnitude and direction of bias depend on the nature and structure of the measurement error. In some cases, the MSE increases substantially, indicating overestimation of variability**.** In contrast, in others, the error suppresses variance estimates, resulting in underestimation**,** which compromises the reliability of statistical conclusions. The absolute differences in MSE between models with and without measurement error serve as a quantitative indicator of the distortion introduced by contamination of the auxiliary variable**.**

Furthermore, the efficiency comparison of estimators (from equations 41 to 48) confirms that all proposed estimators suffer a loss in efficiency under measurement error, with larger MSE values compared to their error-free counterparts. This highlights the sensitivity of variance estimation to data quality and raises concerns about the robustness of conventional estimation strategies in practical settings.

These results align with recent studies emphasizing the critical impact of measurement error on inferential accuracy**.** For instance, Singh and Karan (2021) highlight that even minor errors in auxiliary variables can significantly bias variance estimators. Similarly, Khan et al. (2022) and Riaz et al. (2023) demonstrate that failure to account for measurement error leads to invalid confidence intervals and inefficient resource allocation in survey design.

The findings underscore the importance of implementing **r**igorous data collection protocols**,** robust validation mechanisms**,** and training field staff to minimize measurement errors. Future research should incorporate error-in-variables models into variance estimation frameworks, using instrumental variables or corrected score methods (Carroll et al., 2023). Incorporating robust measures (e.g., median, decile mean) of auxiliary variables can reduce sensitivity to outliers and measurement noise (Riaz et al., 2023). Researchers should disclose potential sources of measurement error and conduct sensitivity analyses to assess their impact on results.

REFERENCES

Allen, J. H., Singh, P., & Samarandache, F. (2003). A family of estimators of population mean using multiauxiliary information in presence of measurement errors. *International Journal of Social Economics, 30*(7), 837–848.

Bowley, A. L. (1926). Measurements of precision attained in sampling. *Bulletin of the Institute of International Statistics, 22*, 1–62.

Buonaccorsi, J. P. (2023). *Measurement error: Models, methods, and applications*. CRC Press.

Carroll, R. J., Ruppert, D., Stefanski, L. A., & Crainiceanu, C. M. (2023). *Measurement error in nonlinear models: A modern perspective* (3rd ed.). Chapman & Hall/CRC.

Cochran, W. G. (1940). The estimation of the yields of cereal experiments by sampling for the ratio gain to total produce. *Journal of Agricultural Society, 30*, 262–275.

Cochran, W. G. (1968). Errors of measurement in statistics. *Technometrics, 10*, 637–666.

Das, A. K., & Tripathi, T. P. (1978). Use of auxiliary information for estimating the finite population variance. *Sankhya, 4*, 139–148.

Das, A. K., & Tripathi, T. P. (1980). Sampling strategies for population mean when the coefficient of variation of an auxiliary character is known. *Sankhya, 42*, 76–86.

Diana, G., & Giordan, M. (2012). Finite population variance estimation in presence of measurement errors. *Communications in Statistics – Theory and Methods, 41*, 4302–4314.

Fuller, W. A. (2009). *Measurement error models*. Wiley.

Kadilar, C., & Cingi, H. (2004). Ratio estimators in simple random sampling. *Applied Mathematics and Computation, 151*, 893–902.

Khan, M., Shabbir, J., & Gupta, S. (2022). Exponential-type estimators for population variance under measurement error. *Journal of Statistical Computation and Simulation, 92*(10), 2145–2160.

Koyuncu, N., & Kadilar, C. (2010). On the family of estimators of population mean in stratified sampling. *Pakistan Journal of Statistics, 26*, 427–443.

Kumar, M. R., Singh, K. A., Singh, A., & Smarandache, F. (2011). Some ratio type estimators under measurement errors. *World Applied Sciences Journal, 14*(2), 272–276.

Laplace, P. S. (1820). *A philosophical essay on probabilities* (English translation, 1951). Dover.

Lohr, S. L. (2022). *Sampling: Design and analysis* (3rd ed.). CRC Press.

Masood, S., & Shabbir, J. (2015). On some families of estimators of post-stratified sample mean using two auxiliary variables. *Communications in Statistics – Theory and Methods, 44*(11), 2398–2415.

Neyman, J. (1934). On the two different aspects of representative method: The method of stratified sampling and the method of purposive selection. *Journal of the Royal Statistical Society, 97*(4), 558–606.

Neyman, J. (1938). Contribution to the theory of sampling human populations. *Journal of the American Statistical Association, 33*(201), 101–116.

Riaz, M., Abbas, N., & Does, R. J. M. M. (2023). Robust estimation of population variance using auxiliary information under non-normality and measurement error. *Quality and Reliability Engineering International, 39*(2), 789–805.

Sharma, P., & Singh, R. (2013). A generalized class of estimator for population variance in presence of measurement errors. *Journal of Modern Applied Statistical Methods, 12*(2), 231–241.

Singh, H. P., & Karpe, N. (2008a). Ratio product estimator for population mean in presence of measurement errors. *Journal of Applied Statistical Sciences, 16*(4), 49–64.

Singh, H. P., & Karpe, N. (2009). Class of estimators using auxiliary information for estimating finite population variance in presence of measurement errors. *Communications in Statistics – Theory and Methods, 38*(5), 734–741.

Singh, H. P., & Karan, P. (2021). Improved estimation of population variance using auxiliary variable under measurement error. *Communications in Statistics – Theory and Methods, 50*(18), 4238–4257.

Singh, R., & Kumar, M. (2011). A family of estimators of population variance using information on auxiliary attribute. In *Studies in Sampling Techniques and Time Series Analysis* (pp. 63–70). Zip Publishing.

Singh, R. P., Chuhan, M., & Sawan, N. (2007). A family of estimators for estimating population mean using known correlation coefficient in two phase sampling. *Statistics in Transition, 8*(1), 89–96.

Solanki, R., Singh, H. P., & Rathore, A. (2012). An alternative estimator for estimating the finite population mean using auxiliary information in sample surveys. *ISRN Probability and Statistics*, Article ID 657682.

Srivastava, A. K., & Jhajj, H. S. (1980). A class of estimators using auxiliary information for estimating finite population variance. *Sankhya, 42*, 87–96.

Srivastava, A. K., & Jhajj, H. S. (1981). A class of estimating of the population mean in survey sampling using auxiliary information. *Biometrika, 68*(1), 341–343.

Watson, D. J. (1937). The estimation of leaf areas. *Journal of Agricultural Sciences, 27*(3), 474–483.

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Appendix A

Let $\bar{z}_h = \bar{Z}_h\left(1+e_{3h}\right)$, $\sigma^2_{xh} = \sigma^2_{xh}\left(1+e_{4h}\right)$, $\sigma^2_{zh} = \sigma^2_{zh}\left(1+e_{5h}\right)$, $\sigma^2_{yh} = \sigma^2_{yh}\left(1+e_{2h}\right)$, $\sigma_{zxh} = \sigma_{zxh}\left(1+e_{7h}\right)$ such that $E\left(e_{ih}\right)=0$ for $i = 0,1,2,3,4,5,7$.

$E\left(e^2_{2h}\right) = \frac{1}{n_h}\left(\lambda_{400h}-1\right)$, $E\left(e^2_{3h}\right) = \frac{1}{n_h}A'_{zh}$, $E\left(e^2_{4h}\right) = \frac{\left(\lambda_{040h}-1\right)}{n_h}$, $E\left(e^2_{5h}\right) = \frac{1}{n_h}A_{zh}$,

$E\left(e^2_{7h}\right) = \frac{1}{n_h}\left(\frac{\lambda_{022h}}{\rho^2_{xzh}}-1\right)$, $E\left(e_{2h}e_{3h}\right) = \frac{1}{n_h}C_{zh}\lambda_{201h}$, $E\left(e_{2h}e_{4h}\right) = \frac{\left(\lambda_{220h}-1\right)}{n_h}$, $E\left(e_{2h}e_{5h}\right) = \frac{1}{n_h}\left(\lambda_{202h}-1\right)$,

$E\left(e_{2h}e_{7h}\right) = \frac{1}{n_h}\left(\frac{\lambda_{211h}}{\rho_{xzh}}-1\right)$, $E\left(e_{3h}e_{4h}\right) = \frac{1}{n_h}C_{zh}\lambda_{021h}$, $E\left(e_{3h}e_{5h}\right) = \frac{1}{n_h}C_{zh}\lambda_{003h}$,

$E\left(e_{3h}e_{7h}\right) = \frac{1}{n_h}C_{zh}\frac{\lambda_{012h}}{\rho_{xzh}}$, $E\left(e_{4h}e_{5h}\right) = \frac{1}{n_h}\left(\lambda_{022h}-1\right)$, $E\left(e_{4h}e_{7h}\right) = \frac{1}{n_h}\left(\frac{\lambda_{031h}}{\rho_{xzh}}-1\right)$,

$E\left(e_{5h}e_{7h}\right) = \frac{1}{n_h}\left(\frac{\lambda_{013h}}{\rho_{xzh}}-1\right)$.

Appendix B: Descriptive Statistics

Population 1			
$N_1 = 20$	$N_2 = 31$	$N_3 = 13$	$N_4 = 16$
$n_1 = 11$	$n_2 = 18$	$n_3 = 8$	$n_4 = 8$
$\bar{y}_{h1} = 3006.55$	$\bar{Y}_{h2} = 4687.226$	$\bar{Y}_{h3} = 6496.231$	$\bar{Y}_{h4} = 7795.3$
$\bar{X}_{h1} = 65.901$	$\bar{X}_{h2} = 141.903$	$\bar{X}_{h3} = 392.385$	$\bar{X}_{h4} = 749.501$
$\bar{Z}_{h1} = 358.2$	$\bar{Z}_{h2} = 713$	$\bar{Z}_{h3} = 1509.54$	$\bar{Z}_{h4} = 2577.47$
$S^2_{yh1} = 572819.2$	$S^2_{yh2} = 432925.6$	$S^2_{yh3} = 162104.7$	$S^2_{yh4} = 426528.6$
$S^2_{xh1} = 129.358$	$S^2_{xh2} = 1909.357$	$S^2_{xh3} = 5349.256$	$S^2_{xh4} = 30437.33$
$\lambda_{4001} = 3.455$	$\lambda_{4002} = 1.564$	$\lambda_{4003} = 1.985$	$\lambda_{4004}=2.346$
$\lambda_{0401} = 1.550$	$\lambda_{0402} = 3.087$	$\lambda_{0403} = 1.492$	$\lambda_{0404}=1.908$
$\lambda_{0041} = 2.451$	$\lambda_{0042} = 2.382$	$\lambda_{0043} = 1.732$	$\lambda_{0044}=2.739$
$\lambda_{2201} = 1.489$	$\lambda_{2202} = 1.733$	$\lambda_{2203} = 1.560$	$\lambda_{2204}=2.046$

$\lambda_{2021} = 2.589$	$\lambda_{2022} = 1.596$	$\lambda_{2023} = 1.819$	$\lambda_{2024} = 2.408$
$\lambda_{0221} = 1.490$	$\lambda_{0222} = 1.90$	$\lambda_{0223} = 1.435$	$\lambda_{0224} = 2.092$
$\lambda_{0211} = -0.140$	$\lambda_{0212} = 0.394$	$\lambda_{0213} = -0.392$	$\lambda_{0214} = 0.461$
$\lambda_{0121} = -0.385$	$\lambda_{0122} = 0.237$	$\lambda_{0123} = -0.512$	$\lambda_{0124} = 0.462$
$\lambda_{2111} = -1.884$	$\lambda_{2112} = 1.453$	$\lambda_{2113} = 1.642$	$\lambda_{2114} = 2.176$
$\lambda_{2011} = -0.071$	$\lambda_{2012} = -0.069$	$\lambda_{2013} = -0.743$	$\lambda_{2014} = 0.662$
$\lambda_{0311} = 1.388$	$\lambda_{0312} = 2.002$	$\lambda_{0313} = 1.405$	$\lambda_{0314} = 1.930$
$\lambda_{0131} = 1.822$	$\lambda_{0132} = 2.038$	$\lambda_{0133} = 1.541$	$\lambda_{0134} = 2.361$
$\lambda_{0031} = -0.728$	$\lambda_{0032} = 0.122$	$\lambda_{0033} = -0.639$	$\lambda_{0034} = 0.432$
$A_{z1} = 2.009423$	$A_{z2} = 0.084583$	$A_{z3} = 0.880598$	$A_{z4} = 2.039794$
$A'_{z1} = 2.262151$	$A'_{z2} = 0.506876$	$A'_{z3} = 3.808552$	$A'_{z4} = 4.422096$

Population 2		
$N_1 = 106$	$N_2 = 106$	$N_3 = 94$
$N_4 = 171$	$N_5 = 204$	$N_6 = 173$
$n_1 = 18$	$n_2 = 18$	$n_3 = 15$
$n_4 = 28$	$n_5 = 33$	$n_6 = 33$
$\overline{Y}_1 = 1536.773$	$\overline{Y}_2 = 2212.594$	$\overline{Y}_3 = 9384.309$
$\overline{Y}_4 = 5588.012$	$\overline{Y}_5 = 966.956$	$\overline{Y}_6 = 404.399$
$\overline{X}_1 = 24375.594$	$\overline{X}_2 = 27421.698$	$\overline{X}_3 = 72409.947$
$\overline{X}_4 = 74364.678$	$\overline{X}_5 = 26441.716$	$\overline{X}_6 = 9843.827$
$\overline{Z}_1 = 24711.811$	$\overline{Z}_2 = 26840.038$	$\overline{Z}_3 = 72723.755$
$\overline{Z}_4 = 73191.199$	$\overline{Z}_5 = 26833.750$	$\overline{Z}_6 = 9903.301$
$S^2_{y1} = 41281745.739$	$S^2_{y2} = 133437790.643$	$S^2_{y3} = 894457432.753$
$S^2_{y4} = 820445636.470$	$S^2_{y5} = 5710998.614$	$S^2_{y6} = 894440.334$
$S^2_{x1} = 2419565834.986$	$S^2_{x2} = 3301722268.346$	$S^2_{x3} = 25842911894.50$

$S_{x4}^2 = 81569146488.325$ $S_{x5}^2 = 2061412415.921$ $S_{x6}^2 = 3353212772.760$

$S_{z1}^2 = 41281745.739$ $S_{z2}^2 = 133437790.643$ $S_{z3}^2 = 894457432.753$

$S_{z4}^2 = 820445636.470$ $S_{z5}^2 = 5710998.614$ $S_{z6}^2 = 894440.334$

$\lambda_{4001} = 78.592$ $\lambda_{4002} = 95.170$ $\lambda_{4003} = 25.540$

$\lambda_{4004} = 101.383$ $\lambda_{4005} = 54.781$ $\lambda_{4006} = 29.946$

$\lambda_{0401} = 27.197$ $\lambda_{0402} = 35.568$ $\lambda_{0403} = 27.408$

$\lambda_{0404} = 97.164$ $\lambda_{0405} = 26.629$ $\lambda_{0406} = 30.089$

$\lambda_{0041} = 27.197$ $\lambda_{0042} = 35.568$ $\lambda_{0043} = 27.408$

$\lambda_{0044} = 97.164$ $\lambda_{0045} = 29.629$ $\lambda_{0046} = 30.089$

$\lambda_{2201} = 32.986$ $\lambda_{2202} = 56.855$ $\lambda_{2203} = 20.582$

$\lambda_{2204} = 98.945$ $\lambda_{2205} = 20.989$ $\lambda_{2206} = 22.947$

$\lambda_{2021} = 33.003$ $\lambda_{2022} = 53.483$ $\lambda_{2023} = 20.243$

$\lambda_{2024} = 92.991$ $\lambda_{2025} = 21.060$ $\lambda_{2026} = 21.721$

$\lambda_{0221} = 27.202$ $\lambda_{0222} = 33.552$ $\lambda_{0223} = 27.254$

$\lambda_{0224} = 91.426$ $\lambda_{0225} = 29.596$ $\lambda_{0226} = 30.728$

$\lambda_{0211} = 4.587$ $\lambda_{0212} = 4.903$ $\lambda_{0213} = 4.499$

$\lambda_{0214} = 8.790$ $\lambda_{0215} = 4.599$ $\lambda_{0216} = 4.691$

$\lambda_{2011} = 5.591$ $\lambda_{2012} = 7.176$ $\lambda_{2013} = 4.004$

$\lambda_{2014} = 8.991$ $\lambda_{2015} = 4.031$ $\lambda_{2016} = 3.920$

$\lambda_{0121} = 4.587$ $\lambda_{0122} = 4.785$ $\lambda_{0123} = 4.488$

$\lambda_{0124} = 8.577$ $\lambda_{0125} = 4.585$ $\lambda_{0126} = 4.709$

$\lambda_{2111} = 32.995$ $\lambda_{2112} = 55.143$ $\lambda_{2113} = 20.507$

$\lambda_{2114} = 95.907$ $\lambda_{2115} = 21.022$ $\lambda_{2116} = 22.295$

$\lambda_{0311} = 27.199$ $\lambda_{0312} = 34.542$ $\lambda_{0313} = 27.330$

$\lambda_{0314} = 94.145$ $\lambda_{0315} = 29.610$ $\lambda_{0316} = 30.372$

$\lambda_{0131} = 27.205$ $\lambda_{0132} = 32.598$ $\lambda_{0133} = 27.181$

$\lambda_{0134} = 88.465$ $\lambda_{0135} = 29.525$ $\lambda_{0136} = 31.161$

$\lambda_{0031} = 4.586$ $\lambda_{0032} = 4.675$ $\lambda_{0033} = 4.478$

$\lambda_{0034} = 8.379$ $\lambda_{0035} = 4.577$ $\lambda_{0036} = 4.754$

$\rho_{zx1} = 0.999$ $\rho_{zx2} = 0.999$ $\rho_{zx3} = 0.999$

$\rho_{zx4} = 0.998$ $\rho_{zx5} = 0.996$ $\rho_{zx6} = 0.993$

$A_{z1} = 61.867$ $A_{z2} = 64.014$ $A_{z3} = 60.688$

$A_{z4} = 175.191$ $A_{z5} = 69.580$ $A_{z6} = 72.980$

$A'_{z1} = 931.800$ $A'_{z2} = 1034.072$ $A'_{z3} = 3806.294$

$A'_{z4} = 5513.881$ $A'_{z5} = 383.8584$ $A'_{z6} = 220.5154$