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Enhancing the Efficiency of CUSUM-Based Location Control Charts Using Robust Statistical Measures

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ABSTRACT

Statistical Quality Control (SQC) involves monitoring and observing goods/industrial procedures to ensure the enhanced quality of goods. Control charts are the primary tools widely valued by businesses for maintaining process consistency. The best standard classes of control charts are the CUSUM charts, also known as the cumulative sum chart. In this work, we propose a novel configuration for CUSUM charts that centers on the use of auxiliary information by a select group of estimators. It involves collective preparation to practice conventional location measures, aiming to improve ratio estimators by utilizing information on auxiliary variables. We have proposed a group of ratio estimators for the population mean in a limited population, utilizing information on auxiliary variables, and employing both conventional and non-conventional measures of location. We have combined the tri-mean, Hodges-Lehmann, mid-range, and decile mean of the auxiliary variables to assist in the purpose. The characteristics related to the suggested group of ratio estimators are estimated using the mean square error. Furthermore, robustness to extreme observations (outliers) is an additional characteristic of the suggested estimators.

Keywords: *Auxiliary information; control chart; average run length; cumulative sum; statistical quality control*

INTRODUCTION

The majority of production variations are linked to the goods in all production and non-production processes. For instance, when a bottle of vegetable ghee is filled, the amount of ghee in the two bottles is different. The lengths and diameters of the two lights do not match when making tubular lights, etc. To improve conditional ARL stability and inference reliability, Li (2022) proposed an adaptive CUSUM with a cautious learning scheme that enhances in-control mean and variance estimates during Phase II monitoring. In a closely related endeavor, Zaman et al. (2022) used score functions (such as Huber's and bi-square weights) and generalized likelihood ratio statistics to create adaptive memory CUSUM models. These methods improve the detection of location and dispersion shifts over designated ranges by dynamically updating reference parameters. Additionally, robust estimation methods have gained popularity. Using median absolute deviation (MAD) estimators based on trimming and winsorization, Khalil et al. (2024) presented a robust CUSUM chart for tracking dispersion. Simulation results demonstrated improved performance under both normal and contaminated data, as measured by Average Run Length (ARL) and Standard Deviation of Run Length (SDRL). NSGA II is used in a novel work on economic statistical design (Sandeep & Mukhopadhyay, 2024) to optimize CUSUM design parameters (sample size, decision limits, and cost) in order to balance cost-efficiency and detection performance. Multivariate and risk-adjusted designs are examples of further extensions: A risk-adjusted multivariate CUSUM chart with Tukey's estimator and accelerated failure time models was proposed by Kazemi et al. (2021) for reliable healthcare process monitoring that is more resilient to skewed or non-normal data.NSGA II is used in a novel work on economic statistical design (Sandeep & Mukhopadhyay, 2024) to optimize CUSUM design parameters (sample size, decision limits, and cost) in order to balance cost-efficiency and detection performance Commonplace (uncontrollable) cause variation is the term used to describe the distinctive measure of the system. Outside, this variance is the leading cause of the unique (controllable) design. The majority of production variations are linked to the goods in all production and non-production processes. For instance, when a bottle of vegetable ghee is filled, the amount of ghee in the two bottles is different. The lengths and diameters of the two lights do not match when making tubular lights, etc. Commonplace (uncontrollable) because variation is the term used to describe the distinctive measure of the system. Outside, this variance is the leading cause of the unique (controllable) design. One of the key ways SQC is implemented is through the use of co7lntrol charts. The three aspects of line quality upper control limits (UCL), center line (CL), and lower control limit (LCL) are represented graphically below. The parameters of the control charts are primarily the two control limits, specifically the LCL (Lower Control Limit) and UCL (Upper Control Limit). This limit was selected such that it is unlikely to be exceeded by the control data. This chance, represented by (α), is commonly known as a False Alarm Rate (FAR) in the literature on quality management.

Shewhart control charts, which contain R and $\overline{X}$ control charts, are also referred to as variable control charts for measurement data. Other characteristics of control charts include p-charts, c-charts, np, and u-charts. According to Shewhart (1924), the Shewhart control chart's structure was created to use only this data and disregard all earlier data; as a result, the charts are not particularly effective and are unable to identify even the most minor changes in processes. Because of this drawback of Shewhart control charts, Page (1954) proposed CUSUM control charts, and Roberts (1959) proposed EWMA. The foundation of creating a control chart is the integration of historical data with current data to enhance the control chart's ability to identify subtle process changes. Performance and average run length (ARL) are the two most frequently used performance metrics in control charts. The ARL is the average number of samples required for the control charts to produce a controlled signal.

In contrast, the performance of the control charts (represented by 1 - β) represents the likelihood of identifying slight changes in the process. Examples of both ARLs are ARLo and ARL1. Since they match the presumption of geometrical distance variables for control charts, Shewhart control charts, such as (X,) R, and S^2, are computed by computing the inverse of power in ARLs. EWMA and CUSUM are examples of Markov methods. ARL can also be calculated by modeling Monte Carlo mileage characteristics or by using the Morkov process attributes to produce the relevant ARL score. Zi, Zou, Zhou, and Wang (2013) and Chen, Cheng, and Xie (2004) both discuss EWMA enhancements. In an unidentified area, we investigated the novel ideas of CUSUM in Chowdhury, Mukherjee, and Chakraborti (2015) and Mukherjee and Marozzi (2016). Brook and Evans (1972), Yeh, Lin, and Venkataramani

(2004), Mukherjee, Graham, and Chakraborti (2013), Graham, Chakraborti, and Mukherjee (2014), and Van Zyl (2015) are only a few of the works that have improved the way CUSUM charts are presented.

Several variables provide insight into every population unit. Interest and auxiliary variables are the two categories into which this variable is separated. The first is known as a study variable and represents a direct interest in research. Later on, it is employed to enhance the sampling strategy or to assess the study variables more accurately. Auxiliary variables are typically linked to the research variables and are utilized in the creation of regression, product, and ratio estimators. A variety of sources, including previous research, economic reports, the national census, and others, can provide information about the auxiliary variables. Because of its strong association with the research variable, auxiliary information is crucial in sample surveys for increasing the accuracy of estimators during the design, estimation, or both phases. Regression, product, and ratio are all helpful estimation techniques. For example, Singh and Tailor (2003), Singh et al. (2004), (2007), Kadilar and Cingi (2006), Gupta and Shabbir (2008), Haq and Shabbir (2013), Singh and Solanki (2013), Yadav and Kadilar (2013), Kadilar (2016), Vishwakarma et al. (2016), Irfan et al. (2018), Surya et al. (2018), and many others have worked to modify sample mean estimators or investigate new families of estimators for the estimation of finite population mean. The auxiliary variables that are helpful in increasing the efficiency of estimators are measured in the literature using the mean, standard deviation, coefficient of skewness, and coefficient of kurtosis. Investigating whether nonconventional measures of auxiliary variables can be used interchangeably with traditional measures is a novel idea. Quartile deviation, mid-range, interquartile range, quartile average, tri-mean, Hodges-lehmann estimators, and many more are examples of non-traditional metrics.

Following the contributions of Tatum (1996), Cook, Zobel, and Nottingham (2004), Sukthomya and Tannock (2005), Kourti (2006), and Raiz (2008a), the concept of utilizing it during the estimation phase in statistical quality control has gained prominence. Raiz utilized process variability and location for monitoring auxiliary control charts, both of which depend on regression-type estimators. Subsequently, Raiz and Does (2009) introduced a variability chart based on ratio-type estimators, demonstrating its superiority over charts reliant on regression-type estimators. Raiz focuses on Shewhart-type charts, which are effective for identifying significant shifts in process parameters. This research proposes methods for efficiently detecting minor shifts in location parameters. Moreover, minimal research has been conducted on incorporating auxiliary information into the structure of the CUSUM chart for monitoring the location parameter. This research presents several CUSUM-type control charts that utilize auxiliary information. The proposed charts' performance is evaluated using average run length (ARL), extra quadratic loss (EQL), ratio of average run length (RARL), and performance comparison index (PCI).

The structure of the remainder of this article is as follows: Section 2 presents the design structure of the classical CUSUM control chart. Section 3 outlines the specifics of the proposed CUSUM charts and their associated performance metrics. Section 4 presents a comparison between the proposed charts and typical CUSUM control charts. Section 5 delineates the implementation steps and illustrates their applicability through an example utilizing simulated data sets. Section 6 ends the findings of this article.

CLASSICAL CUSUM CONTROL CHART

Currently, the CUSUM chart, proposed by Page in 1954, is the most widely favored algorithm for monitoring and controlling production processes in industries. The establishment of the classical control chart and the Sequential Probability Ratio Test (SPRT) by Wald in 1947 is closely related. Fuh (2003) also investigated this link and concluded that both CUSUM and SPRT constituted a concealed model of a

Markov chain. The two-sided CUSUM chart features two statistics, R_i^+ and R_i^-, in addition to a single control limit, H.

$$R_i^+ = \max[0, (\overline{Y}_i - \mu_0) - K + R_{i-1}^+]$$

$$R_i^- = \max[0, -(\overline{Y}_i - \mu_0) - K + R_{i-1}^-]$$

Based on Irfan et al. (2018), represent the sample number. The study variable Y has a sample mean designated as $\overline{Y}$, and the goal mean of the research variable is μ_0. The Cusum has a reference value K, which is defined as half of the shift value (Ewan & Kemp, 1960). For the two plotting statistics, the initial value is set to zero, i.e., R_i^+ = R_i^- = 0. The two statistics are evaluated against the control limit H, determining that if R_i^+ exceeds H for any value of i, the process mean has increased.

In contrast, if R_i^- exceeds H for any value of i, the process mean has decreased. The CUSUM chart consists of two parameters, denoted as K and H. The parameters of the CUSUM chart are meticulously selected due to the sensitivity of the Average Run Length (ARL). The conventional technique employs these two parameters (Montgomery, 2009) as follows:

$$= k \times \sqrt{Var(\overline{Y})} \text{and} H = h \times \sqrt{Var(\overline{Y})}$$

As $(\overline{Y}) = {}^{\sigma_Y}/_{\sqrt{n}}$, study variable Y has standard deviation σ_Y.

PROPOSED CUSUM CONTROL CHARTS

Let (y_{i1}, x_{i1}), (y_{i2}, x_{i2}), (y_{i3}, x_{i3}) ,……., (y_{in}, x_{in}) (where $i = 1, 2, ...$) represent a sequence of paired observations for the quality characteristic Y (the study variable), which is correlated with an auxiliary variable X. Each pair (Y_{ij}, X_{ij}) for $j = 1, 2, 3, ..., n$ is assumed to follow a bivariate normal distribution characterized by a mean vector and a covariance matrix, expressed as:

$$\mu = \begin{pmatrix} \mu_0 + \delta\sigma_Y \\ \mu_X \end{pmatrix}, \ \Sigma = \begin{pmatrix} \sigma_Y^2 & Cov(Y, X) \\ Cov(X, Y) & \sigma_X^2 \end{pmatrix}$$

Where μ_0 is the in-control mean of the study variable Y and μ_X is the known mean of the auxiliary variable X. In this context, σ_Y^2 and σ_X^2 are the population variances of Y and X, respectively, and are assumed to be known. $Cov(Y, X) = Cov(X, Y)$ is the covariance between the study variable Y and the auxiliary variable X. in addition, δ represents the amount of shift introduced in the study variable Y in σ_Y units i.e. $\delta = \frac{|\mu_1 - \mu_0|}{\sigma_Y}$, where μ_1 represents the out-of-control mean of Y.

Table 1: Some estimators for estimating population mean

Estimators ($M_j, j = 1, 2, \ldots, 5$)	E (M_j)	MSE
$M_1 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$	$\bar{Y}$	$\phi \bar{Y}^2 C_y^2$
$M_2 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right), \bar{x} \neq 0$	$\bar{Y}$	$\phi \bar{Y}^2 [C_y^2 + C_x^2 - 2\rho_{yx} C_y C_x]$
$M_3 = \bar{y} + b(\bar{X} - \bar{x})$	$\bar{Y}$	$\phi \hat{\bar{Y}}^2 C_y^2 [1 - \rho_{yx}^2]$
$M_4 = \bar{y} \exp\left(\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right)$	$\bar{Y}$	$\frac{\phi}{4} \bar{Y}^2 [4C_y^2 + C_x^2 - 4\rho_{yx} C_y C_x]$
$M_5 = \left[\frac{\bar{y}}{2}\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right\} + h_1(\bar{X}-\bar{x}) + h_2\bar{y}\right] \exp\left(\frac{Q_D(\bar{X}-\bar{x})}{Q_D(\bar{X}+\bar{x}) - 2T_M}\right)$	$\bar{Y}$	$\frac{1}{4}\phi C_x^2\left[4\left(\bar{Y}\theta + \bar{X}h_1\right)^2 + 4\bar{Y}\left\{\left(5\theta^2 + \frac{1}{4}\right)\bar{Y} + 4\bar{X}h_1\theta\right\}h_2 + 16\bar{Y}^2\theta^2 h_2^2\right] + \bar{Y}^2\left[h_2^2 + \phi C_y^2 (1+h_2)^2\right] - 2\bar{Y}\phi\rho_{yx} C_y C_x\left[(1+h_2)\left(\bar{Y}\theta + \bar{X}h_1 + 2\bar{Y}\theta h_2\right)\right]$
$M_6 = \left[\frac{\bar{y}}{2}\left\{\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) + \exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right)\right\} + h_1(\bar{X}-\bar{x}) + h_2\bar{y}\right] \exp\left(\frac{Q_D(\bar{X}-\bar{x})}{Q_D(\bar{X}+\bar{x}) - 2M_R}\right)$	$\bar{Y}$	$\frac{1}{4}\phi C_x^2\left[4\left(\bar{Y}\theta + \bar{X}h_1\right)^2 + 4\bar{Y}\left\{\left(5\theta^2 + \frac{1}{4}\right)\bar{Y} + 4\bar{X}h_1\theta\right\}h_2 + 16\bar{Y}^2\theta^2 h_2^2\right] + \bar{Y}^2\left[h_2^2 + \phi C_y^2 (1+h_2)^2\right] - 2\bar{Y}\phi\rho_{yx} C_y C_x\left[(1+h_2)\left(\bar{Y}\theta + \bar{X}h_1 + 2\bar{Y}\theta h_2\right)\right]$

Irfan et al. (2018) identify various estimators in the literature for estimating the population mean. Robust estimators are chosen for their superior efficiency compared to the simple mean estimator. Table 1 presents the chosen estimator, its expected value, and the Mean Squared Error (MSE). An estimator exhibiting a lower mean squared error (MSE) is considered more efficient than its alternative. The MSEs of the chosen estimator are typically lower than those of the simple mean estimator, contingent upon the correlation between the variables X and Y. Further statistical insights and characteristics of the chosen estimators are available in the works of Cochran (1940), Murthy (1964), Watson (1937), and Bahl and Tuteja (1991).

Some of the quantities in Table 1 are defined as: $e_0 = \frac{\bar{y} - \bar{Y}}{\bar{Y}}$ and $e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}$ Such that

$E(e_0) = 0$; $E(e_1) = 0$; $E(e_0^2) = \phi C_y^2$; $E(e_1^2) = \phi C_x^2$; $E(e_0 e_1) = \phi \rho_{yx} C_y C_x$

Where $\phi = \frac{1}{n} - \frac{1}{N}$, Coefficient of variation: $C_y^2 = (\bar{Y}^2)^{-1} S_y^2$; $C_x^2 = (\bar{X}^2)^{-1} S_x^2$, Coefficient of correlation: $\rho_{yx} = (S_y S_x)^{-1} S_{yx}$. , Quartile deviation: $Q_D = \frac{Q_3 - Q_1}{2}$, Mid-range: $M_R = \frac{x_{(1)} + x_{(N)}}{2}$, Tri-mean:

$$T_M = \frac{Q_1 + 2Q_2 + Q_3}{4}$$

$$h_1 = \frac{\bar{Y}\left[-2\rho_{yx} C_y + C_x \left\{2\vartheta - 2\phi\vartheta C_x^2 \left(\vartheta^2 + \frac{1}{4}\right) + \phi\rho_{yx} C_y C_x \left(\vartheta^2 + \frac{1}{4}\right) + 2\vartheta\phi \left(-1 + \rho_{yx}^2\right) C_y^2\right\}\right]}{2\bar{X} C_x \left[-1 + \phi C_y^2 \left(-1 + \rho_{yx}^2\right)\right]}$$ and

$h_2 = \frac{\phi\left[C_x^2\left(\vartheta^2 + \frac{1}{4}\right) - 2C_y^2\left(-1+\rho_{yx}^2\right)\right]}{4\left[-1+\phi C_y^2\left(-1+\rho_{yx}^2\right)\right]}$ are the optimal values of h_1 and h_2 that minimizes the mean square error.

This paper use this efficiency (see Table 1) to develop CUSUM-type structures and examines the impact of these efficient estimators on several performance metrics (ARL, EQL, RARL, and PCI) of CUSUM charts.

The plotting statistics for the proposed CUSUM charts (based on the estimators given in Table 1) are given as:

The plotting statistics for the proposed CUSUM charts (based on the estimators given in Table 1) are given as:

$$R_i^+ = \max\left[0, (\overline{Y}_i - \mu_0) - K + R_{i-1}^+\right]$$
$$R_i^- = \max\left[0, -(\overline{Y}_i - \mu_0) - K + R_{i-1}^-\right]$$

Starting values for (4) are taken equal to zero, i.e., the statistics and are plotted against the control limit *Hj*. The decision rule for the proposed CUSUM chart is given as: for any value of *i*, if the value of exceeds the value of *Hj*, then the process mean is declared to have shifted upward, and if the value of exceeds the value of *H*, then the process mean is declared to have moved downward. The values H and Hj are defined as:

$K_j = k_j \times \sqrt{MSE(M_j)}andH_j = h_j \times \sqrt{MSE(M_j)}$

i. Where are the design parameters of the proposed CUSUM charts with an estimator? For more details about the selection of and, see Gan (1991). The values of and need to be selected very carefully because the properties of the proposed CUSUM chart mainly depend on these two constants (along with the values of). For some selected values of and, the corresponding values are found by running simulations in R software (R Core Team, 2014) such that the is fixed at 371. Based on the constants, the values of the proposed CUSUM chart for all the estimators are given in Table 2-10.
ii. From tables (2-10), we present the chief findings of our proposed CUSUM control charts.The use of auxiliary variable along with robust measures improves the performance of CUSUM chart, especially when the value of ρ_{YX} is reasonably large.
iii. For a fixed value of ARL_0, the ARL_1 values decrease rapidly with increase in the values of either or both ρ_{YX} and $|\delta|$.
iv. For $n=5$ and different levels of correlation coefficient and shift i.e. $\rho_{YX}(0.30, 0.60, and\, 0.90)$ and $\delta = (0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00, 3.00, 4.00)$ it is observed that the estimators having both auxiliary variable along with robust measures M_5 and M_6 gives rapid decrease in average run length by increasing correlation coefficient and shift.

v. For a correlation coefficient and shift at different levels, as mentioned in the above point. The estimators give a decrease in values as compared to other estimators.
vi. For along with the combinations of different values of and. Robust estimators give better results (i.e., rapid decrease in values).
vii. It is observed that by increasing sample size, correlation coefficient, and shifting, the robust estimators give the best results as compared to other estimators.
viii. From the findings stated above, it is inferred that the proposed CUSUM charts with a robust estimator show better performance in different situations.
ix. The findings in points are based on.

COMPARISONS

It is generally used to compare the performance of different charts. Wu, Jiao, Yang, Liu, and Wang (2009) highlighted some of the drawbacks of. One of the drawbacks is that it only gives the performance of a control chart for a specific shift size. Hence, they recommended some measures that evaluate the performance of a control chart over a range of values. These measures are named as Extra Quadratic Loss (EQL) and Ratio of Average Run Lengths (RARL), which are defined as:

Table 2: ARL **values at** $ARL_0 = 371$, $n = 5$ **and** $\rho_{yx} = 0.3$

n	ρ_{yx}	0.30					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
5	0.25	370.37	371.87	371.05	371.16	371.04	368.63
	0.50	28.23	32.05	26.33	34.39	35.42	23.34
	0.75	8.40	10.81	7.98	9.13	9.12	7.60
	1.00	4.80	6.11	4.65	4.99	5.04	4.48
	1.25	3.37	4.27	3.31	3.46	3.49	3.17
	1.50	2.67	3.34	2.59	2.73	2.75	2.51
	1.75	2.24	2.75	2.24	2.31	2.28	2.16
	2.00	1.96	2.34	1.96	2.04	2.01	1.92
	3.00	1.07	1.50	1.05	1.12	1.07	1.04

Table 3: ARL ***values at*** $ARL_0 = 371$, $n = 5$ ***and*** $\rho_{yx} = 0.6$

n	ρ_{yx}	0.60					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
5	0.25	370.37	371.97	370.56	372.48	371.05	372.28
	0.50	28.23	20.69	22.67	22.32	22.21	17.01
	0.75	8.40	7.06	7.07	6.71	6.70	6.00
	1.00	4.80	4.25	4.15	3.87	3.86	3.65
	1.25	3.37	3.08	3.01	2.80	2.85	2.69
	1.50	2.67	2.43	2.40	2.29	2.25	2.18
	1.75	2.24	2.08	2.07	1.98	1.97	1.90
	2.00	1.96	1.84	1.85	1.76	1.72	1.65
	3.00	1.07	1.05	1.01	1.00	1.00	1.00

Table 4: ARL **values at** $ARL_0 = 371$, $n = 5$ **and** $\rho_{yx} = 0.9$

n	ρ_{yx}	0.90					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
5	0.25	370.37	370.23	371.97	372.14	372.60	371.06
	0.50	28.23	7.02	9.77	7.58	7.25	6.72
	0.75	8.40	3.02	3.68	3.08	3.00	2.93
	1.00	4.80	2.05	2.38	2.09	2.06	2.00
	1.25	3.37	1.61	1.94	1.68	1.59	1.55
	1.50	2.67	1.23	1.50	1.22	1.15	1.16
	1.75	2.24	1.03	1.12	1.04	1.01	1.01
	2.00	1.96	1.00	1.01	1.00	1.00	1.00
	3.00	1.07	1.00	1.00	1.00	1.00	1.00

Table 5: ARL ***values at*** $ARL_0 = 371$, $n = 10$ ***and*** $\rho_{yx} = 0.3$

n	ρ_{yx}	0.30					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
10	0.25	371.09	371.52	370.26	373.90	373.90	371.36
	0.50	14.79	19.50	13.50	16.01	15.80	13.31
	0.75	5.12	6.65	4.82	5.22	5.31	4.81
	1.00	3.13	3.93	2.98	3.15	3.17	3.00
	1.25	2.33	2.88	2.27	2.38	2.37	2.27
	1.50	1.96	2.29	1.92	1.94	1.95	1.92
	1.75	1.70	1.94	1.63	1.70	1.69	1.63
	2.00	1.40	1.76	1.28	1.39	1.39	1.29
	3.00	1.00	1.01	1.00	1.00	1.00	1.00

Table 6: ARL ***values at*** $ARL_0 = 371$, $n = 10$ ***and*** $\rho_{yx} = 0.6$

n	ρ_{yx}	0.60					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
10	0.25	371.09	371.52	370.60	369.51	369.54	369.54
	0.50	14.79	12.07	10.58	11.13	11.40	10.17
	0.75	5.12	4.57	4.02	4.10	4.11	3.86
	1.00	3.13	2.89	2.61	2.62	2.61	2.54
	1.25	2.33	2.16	1.99	2.01	1.98	1.94
	1.50	1.96	1.81	1.69	1.72	1.70	1.65
	1.75	1.70	1.58	1.29	1.32	1.31	1.27
	2.00	1.40	1.25	1.05	1.07	1.04	1.04
	3.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 7: ARL ***values at*** $ARL_0 = 371$, $n = 10$ ***and*** $\rho_{yx} = 0.9$

n	ρ_{yx}	0.90					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
10	0.25	371.09	371.52	370.26	371.53	371.52	372.92
	0.50	14.79	4.55	4.67	4.58	4.56	4.37
	0.75	5.12	2.16	2.20	2.13	2.15	2.12
	1.00	3.13	1.56	1.56	1.51	1.51	1.50
	1.25	2.33	1.04	1.05	1.02	1.02	1.02
	1.50	1.96	1.00	1.00	1.00	1.00	1.00
	1.75	1.70	1.00	1.00	1.00	1.00	1.00
	2.00	1.40	1.00	1.00	1.00	1.00	1.00
	3.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 8: ARL ***values at*** $ARL_0 = 371$, $n = 15$ ***and*** $\rho_{yx} = 0.3$

n	ρ_{yx}	0.30					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
15	0.25	371.14	371.29	370.78	371.01	371.61	370.78
	0.50	10.18	13.03	9.11	10.09	10.04	9.11
	0.75	4.02	4.85	3.69	3.90	3.88	3.69
	1.00	2.57	3.13	2.43	2.49	2.50	2.43
	1.25	1.97	2.35	1.91	1.94	1.96	1.92
	1.50	1.63	1.93	1.53	1.62	1.60	1.54
	1.75	1.29	1.69	1.18	1.22	1.22	1.19
	2.00	1.05	1.41	1.03	1.04	1.03	1.02
	3.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 9: ARL ***values at*** $ARL_0 = 371$, $n = 15$ ***and*** $\rho_{yx} = 0.6$

n	ρ_{yx}	0.60					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
15	0.25	371.14	370.78	371.61	370.78	370.78	371.43
	0.50	10.18	8.52	7.28	7.58	7.56	7.10
	0.75	4.02	3.45	3.07	3.15	3.16	3.05
	1.00	2.57	2.35	2.10	2.13	2.10	2.07
	1.25	1.97	1.88	1.67	1.70	1.69	1.65
	1.50	1.63	1.49	1.19	1.23	1.21	1.19
	1.75	1.29	1.15	1.02	1.03	1.02	1.01
	2.00	1.05	1.02	1.00	1.00	1.00	1.00
	3.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 10: ARL **values at** $ARL_0 = 371$, $n = 15$ **and** $\rho_{yx} = 0.9$

n	ρ_{yx}	0.90					
	Chart→ δ↓	M_1	M_2	M_3	M_4	M_5	M_6
15	0.25	371.14	371.29	371.60	371.29	370.78	371.60
	0.50	10.18	3.45	3.44	3.45	3.39	3.35
	0.75	4.02	1.87	1.85	1.82	1.83	1.82
	1.00	2.57	1.14	1.10	1.10	1.08	1.08
	1.25	1.97	1.00	1.00	1.00	1.00	1.00
	1.50	1.63	1.00	1.00	1.00	1.00	1.00
	1.75	1.29	1.00	1.00	1.00	1.00	1.00
	2.00	1.05	1.00	1.00	1.00	1.00	1.00
	3.00	1.00	1.00	1.00	1.00	1.00	1.00

Table 11: EQL, RARL and PCI values for ARL=371

N	ρ_{yx}	Measures	M_1	M_2	M_3	M_4	M_5	M_6
5	0.3	EQL	7.0223	8.7625	6.9464	7.2528	7.1635	6.7950
		RARL	1.0529	1.3076	1.0337	1.1039	1.0967	1.0000
		PCI	1.0335	1.2895	1.0223	1.0674	1.0542	1.0000
	0.6	EQL	7.0223	6.7436	6.5867	6.4096	6.3499	6.2116
		RARL	1.2150	1.1204	1.1025	1.0613	1.0508	1.0000
		PCI	1.1305	1.0857	1.0604	1.0319	1.0223	1.0000
	0.9	EQL	7.0223	5.5005	5.5948	5.5099	5.4896	5.4828
		RARL	1.8188	1.0136	1.0978	1.0244	1.0095	1.0000
		PCI	1.2808	1.0032	1.0204	1.0049	1.0012	1.0000
10	0.3	EQL	5.9332	6.4601	5.8539	5.9345	5.9266	5.8557
		RARL	1.0299	1.1972	1.0000	1.0366	1.0351	1.0000
		PCI	1.0135	1.1035	1.0000	1.0138	1.0124	1.0003
	0.6	EQL	5.9332	5.8109	5.6671	5.6830	5.6704	5.6463
		RARL	1.1412	1.0780	1.0123	1.0215	1.0185	1.0000
		PCI	1.0508	1.0292	1.0037	1.0065	1.0043	1.0000
	0.9	EQL	5.9332	5.3979	5.3996	5.3932	5.3946	5.3948
		RARL	1.5249	1.0074	1.0108	1.0000	1.0036	1.0039
		PCI	1.1001	1.0009	1.0012	1.0000	1.0002	1.0003
15	0.3	EQL	5.6545	5.9299	5.6074	5.6348	5.6321	5.6085
		RARL	1.0296	1.1579	1.0000	1.0192	1.0181	1.0007
		PCI	1.0084	1.0575	1.0000	1.0049	1.0044	1.0002
	0.6	EQL	5.6545	5.5866	5.5025	5.5132	5.5082	5.4978
		RARL	1.1177	1.0635	1.0043	1.0133	1.0103	1.0000
		PCI	1.0285	1.0162	1.0009	1.0028	1.0019	1.0000
	0.9	EQL	5.6545	5.3718	5.3701	5.3696	5.3689	5.3686
		RARL	1.4104	1.0070	1.0039	1.0030	1.0010	1.0000
		PCI	1.0533	1.0006	1.0003	1.0002	1.0001	1.0000

$$EQL = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{\delta_{\min}}^{\delta_{\max}} \delta^2 ARL(\delta) d\delta$$

$$RARL = \frac{1}{\delta_{max} - \delta_{min}} \int_{\delta_{min}}^{\delta_{max}} \frac{ARL(\delta)}{ARl_{benchmark}(\delta)} d\delta$$

Another performance measure named as PCI given by Ou, Wen, Wu, and Khoo (2012) is defined as:

$$PCI = \frac{EQL}{EQL_{benchmark}}$$

Where are they evaluated for the benchmark chart? The best chart is taken as the benchmark chart in this study.In the current study, the comparisons of all the charts under discussion in the form of and are provided in Table 11, where the in-control for all the charts is fixed at. From the findings of Table 2-10, it is observed that the proposed CUSUM charts with robust estimators show better performance in different situations, and their performance improves as the correlation coefficient increases. Hence, when and is selected for comparison purposes, it is compared with the traditional CUSUM. In Table 11, a smaller value of indicates better performance of a chart, and accordingly, the best chart (indicated by bold value)in all situations is taken as the benchmark chart in computing and similarly, the value of and are greater than one means that the benchmark chart has a superior overall performance and vice versa. In Table 11, with a different sample size, it gives the overall best performance because it has the lowest EQL value. It can be clearly seen from Table 11 that *RAuxCUSUM is* outperforming the classical CUSUM charts when there is a strong positive correlation between the study variable and the auxiliary variables.

Implementation steps and Illustrative Example

The steps involved in implementing any of the *RAuxCUSUM* are listed below;

1. Specify the values of the in control parameters$\left(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2 and\, \rho_{yx}\right)$.
2. In the case where the parameters are not known, they must expected as of data which are in-control.
3. For every sample being observed, calculate the values of the estimators.
4. As for the precise the shift in procedure mean which is perceived, calculate design parameters (k_p, h_p) values, afterwards calculate control limits R_i^+ and R_i^-.
5. For $(i = 1, 2, \ldots\ldots..)$, plot control limits against h_p. If any these two control limits cross over the value of h_p, the process mean value has lifted rising overhead $(for R_i^+ > h_p)$ and downhill under $(for R_i^- > h_p)$ the in-control mean of Y. Letter the initial limit overpass, as, the lowest value of i for which R_i^+ and R_i^- beats h_p. The i is mentioned as the run length. That is wealth to restate that the suggested two sided system. Though, as one sided upper or lower control chart is of attention, plot R_i^+ and R_i^- in contradiction of h_p

6. At last, the cause of variation among the process mean or target mean is examined and corrective trials are engaged to carry the process hind to an in-control process.

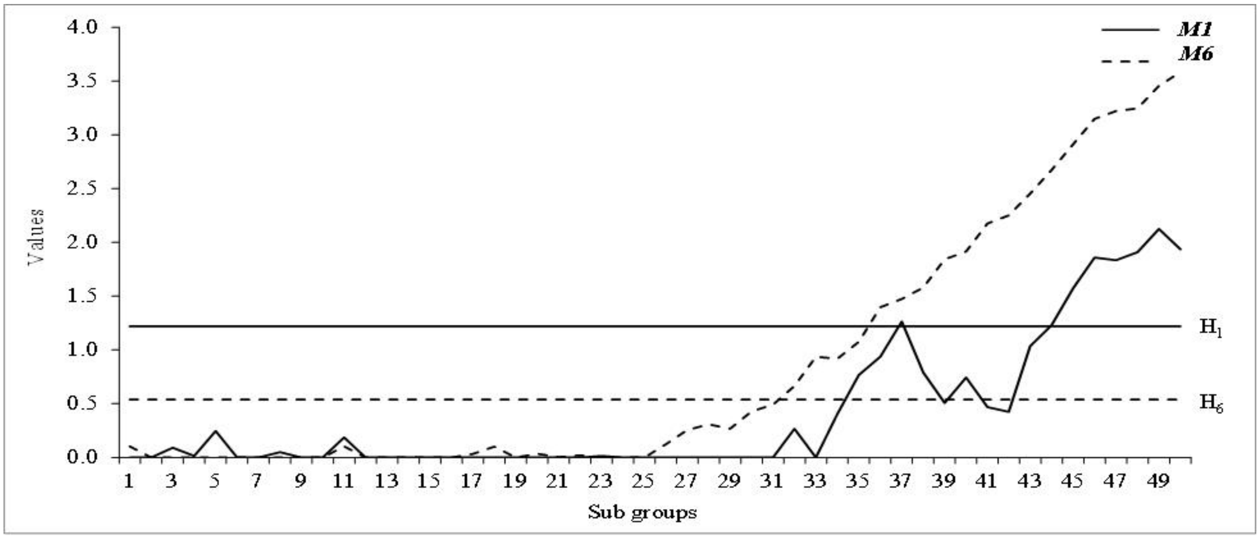

Figure 1(a):Control Charting Display of M_1 and M_6 CUSUM charts

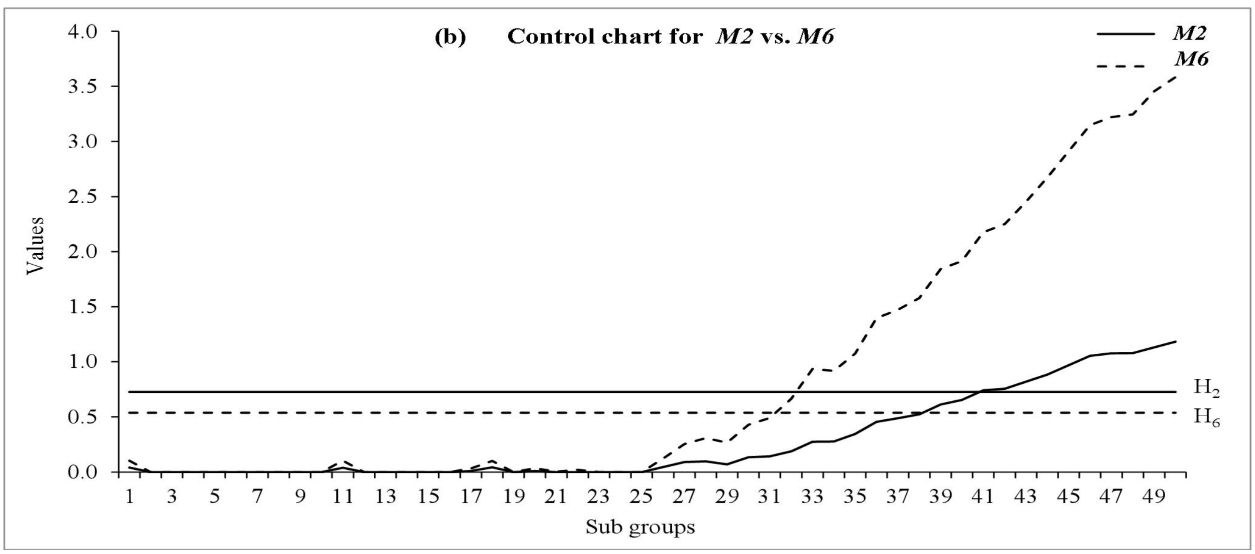

Figure 1(b): Control Charting Display of M_2 and M_6 CUSUM charts

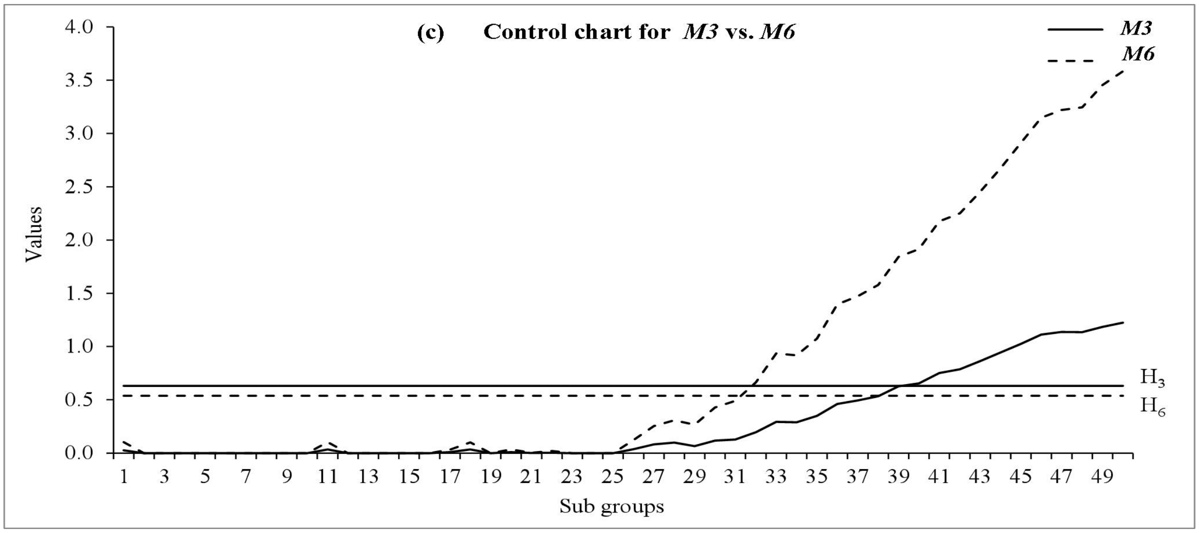

Fig Figure 1(c): Control Charting Display of M_3 and M_6 CUSUM charts

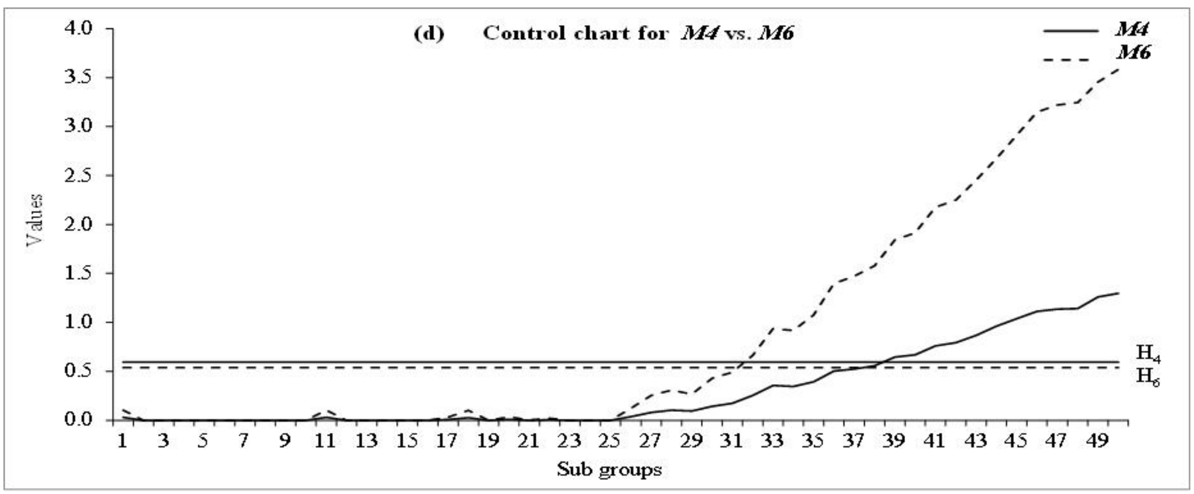

Figure 1(d):Control Charting Display of M_3 and M_6 CUSUM charts

Mansion in Example details that the Control Chart given in Figure 10 is made for the falling choices: $\mu_y = \mu_x = 5$; $\sigma_y = \sigma_x = 1$; Shift $\delta = 0.25$, $n = 15$ and$\rho_{yx} = 0.90$.With the use of auxiliary variable the performance of Cusum chart improved, as the value of the correlation coefficient ρ_{YX} is large. This is from the ARL_1 values. As the ARL is fixed at 370, the corresponding ARL_1 values becomes lesser with the increase of either or both coefficient ρ_{YX} and shift $|\delta|$. h_p ranges for the values of k_p static as (0.25, 0.50, 0.75, 1.00), correspondingly, completed all values of coefficient ρ_{YX} measured. Figure 1(a)shows that the estimator M_1 detected shift in 8 sub-groups while M_6 detected shift in 19 sub-groups. The M_6 chart identified a greater number of out-of-control signals compared to the M_1 chart, indicating that the M_6 chart possesses superior detection capability relative to the M_1 chart. Figure 1(b) illustrates that the estimator M_2 identified shifts in 10 sub-groups, whereas M_6 identified shifts in 19 sub-groups. Figure

1(c) indicates that the estimator M_3 identified shifts in 11 sub-groups, whereas M_6 identified shifts in 19 sub-groups. Figure 1(d) illustrates that the estimator M_4 identified shifts in 12 sub-groups, whereas M_6 identified shifts in 19 sub-groups. The proposed CUSUM charts indicate that robust estimators demonstrate greater efficiency in detecting outliers within the data. Estimators with robust measures are appropriate for atypical data sets. The findings from the illustrative example corroborate those presented in Section 3, which indicate that RAuxCUSUM6 demonstrates the best performance.

SUMMARY, CONCLUSIONS AND RECOMMENDATION

The value of industrial trade and services is a significant concern for the management sector of industry. SQC provides effective tools for monitoring and enhancing product quality, thereby minimizing undesirable variations in production. Control charts are a crucial method of Statistical Quality Control (SQC), categorized into Shewhart, CUSUM, and EWMA control charts. Shewhart charts are designed to detect significant shifts in processes, whereas CUSUM and EWMA charts are aimed at identifying small and moderate shifts. This study introduces new CUSUM-type control charts, referred to as RAuxCUSUM control charts, and designed for monitoring the mean of a process. The proposed charts utilize robust estimators that rely on auxiliary variables. The proposed charts have been examined, revealing that estimators perform optimally under various conditions. The proposed charts are presented using various measures, including ARL, EQL, and PCI. The study revealed that the proposed charts represent a comprehensive application of classical CUSUM charts. The comparative analysis revealed that the proposed structures outperform traditional CUSUM charts. This example aims to evaluate the presentation of proposed CUSUM charts, as robust estimators identify more points than traditional estimators. The proposed CUSUM charts indicate that as the sample size increases, the CUSUM points fall outside the control limits. The analysis concludes that the proposed CUSUM chart demonstrates greater efficiency than the classical CUSUM chart in the presence of outliers or unusual data points. This is attributed to the inclusion of robust estimators, which effectively address the challenges posed by atypical data. Researcher recommends extending this work to simultaneously assess large and small shifts in the system by employing a combined Shewhart-CUSUM method. Additionally, the auxiliary information may be extended for the monitoring of dispersion parameters.

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