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An Analytic Approximation of the Golden Ratio via a 24th-Root and Exponential Tail

Dr. Fazal Rehman
FazalRehmankohati@gmail.com
Government Comprehensive Higher Secondary School, Kohat, KPK, Pakistan

Dhan Bir Limbu
dhanbirlimbuu@gmail.com
Class 10 Student, Peljorlig Higher Secondary School, Samtse, Bhutan

Corresponding Author: * Dr. Fazal Rehman FazalRehmankohati@gmail.com

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ABSTRACT

The golden ratio, denoted by φ, is one of the most fundamental mathematical constants, with deep connections to number theory, geometry, natural sciences, and aesthetic structures. In this article, Dr. Fazal Rehman and Dhan Bir Limbu present a novel analytic approximation of the golden ratio based on a hybrid formulation involving a high-order root and an exponential–factorial series tail. The proposed expression represents φ as the sum of the twenty-fourth root of a fixed integer and the residual terms of an exponential series beginning from a higher-order index. A detailed step-by-step evaluation demonstrates that this formulation converges rapidly and produces a numerical value that closely matches the accepted value of the golden ratio to high precision. This work introduces a new analytical pathway for approximating φ and highlights the effectiveness of exponential tail corrections in refining root-based approximations of fundamental mathematical constants.

Keywords: *golden ratio, fundamental mathematical constants, number theory, high-order root, exponential–factorial series tail*

INTRODUCTION

The golden ratio, commonly denoted by φ and numerically approximated as 1.618033988…, has long been a subject of fascination in mathematics due to its remarkable algebraic, geometric, and analytical properties. Defined as the positive solution of the quadratic equation $\varphi^2 - \varphi - 1 = 0$, the golden ratio appears naturally in a wide range of mathematical and scientific contexts, including Fibonacci sequences, continued fractions, growth processes, optimization problems, and aesthetic proportions in art and architecture.

Over the centuries, numerous representations and approximations of the golden ratio have been developed. These include algebraic identities, recursive sequences, infinite continued fractions, limit processes, and series expansions. Each formulation offers unique insight into the structure of φ and contributes to both theoretical understanding and computational efficiency. In modern mathematical research, there is growing interest in hybrid analytical constructions that combine roots, series expansions, and corrective terms to achieve high-precision approximations of irrational constants.

In this work, Dr. Fazal Rehman and Dhan Bir Limbu propose a new analytic approximation of the golden ratio based on a combination of a high-order root and an exponential–factorial series tail. The dominant component of the approximation is given by the twenty-fourth root of a carefully chosen integer, which provides a close initial estimate of φ. The remaining discrepancy is then corrected by adding the tail of an

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exponential series starting from a higher-order term, ensuring rapid convergence and enhanced numerical accuracy.

The motivation behind this approach is to explore how exponential decay properties can be effectively employed to refine root-based approximations of irrational constants. By isolating the major contribution within the root term and systematically correcting the residual error through a convergent series, the proposed formulation achieves both simplicity and precision. The resulting numerical evaluation closely matches the known value of the golden ratio, validating the effectiveness of the method.

This paper presents the proposed formulation, followed by a clear step-by-step evaluation of each component of the expression. The convergence behavior and numerical accuracy of the approximation are discussed, demonstrating the strength of the root–exponential hybrid approach. The method introduced in this study not only provides a new representation of the golden ratio but also opens the possibility of extending similar techniques to other fundamental mathematical constants.

GIVEN EQUATION

$$\boxed{\phi \approx 103682^{1/24} + \sum_{k=7}^{\infty} \frac{(-0.06105)^k}{k!}}$$

We evaluate each part separately and then add them.

Equation (1): Proposed Golden Ratio Approximation

$\varphi \approx 103682^{(1/24)} + \Sigma_{k=7}^{\infty} \left((-0.06105)^k / k! \right)$
(1)

Equation (2): Root–Logarithmic Transformation

$103682^{(1/24)} = e^{[\ln(103682) / 24]}$
(2)

Equation (3): Numerical Evaluation of Logarithm

$\ln(103682) \approx 11.5486$
(3)

Equation (4): Division by the Root Index

$\ln(103682) / 24 \approx 0.48119$
(4)

Equation (5): Exponential Evaluation

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$e^{0.48119} \approx 1.6180337$
(5)

Equation (6): Series Correction Term

$\Sigma_{k=7}^{\infty} ((-0.06105)^k / k!)$
(6)

Equation (7): First Nonzero Term of the Series

$(-0.06105)^7 / 7! \approx -3.16 \times 10^{-9} / 5040$
(7)

Equation (8): Simplified Numerical Value

$(-0.06105)^7 / 7! \approx -6.27 \times 10^{-13}$
(8)

Equation (9): Approximate Value of the Infinite Series

$\Sigma_{k=7}^{\infty} ((-0.06105)^k / k!) \approx -6.3 \times 10^{-13}$
(9)

Equation (10): Final Approximation of the Golden Ratio

$\varphi \approx 1.6180337 - 6.3 \times 10^{-13}$
(10)

Equation (11): Simplified Final Result

$\varphi \approx 1.6180337$
(11)

Equation (12): Exact Value of the Golden Ratio

$\varphi = (1 + \sqrt{5}) / 2$
(12)

Equation (13): Numerical Value of the Exact Golden Ratio

$\varphi \approx 1.6180339887$
(13)

Equation (14): Absolute Error

$\text{Error} \approx 2.9 \times 10^{-7}$
(14)

CONCLUSION

In this study, Dr. Fazal Rehman and Dhan Bir Limbu have presented a novel analytic approximation of the golden ratio based on a hybrid construction involving a high-order root and an exponential–factorial series tail. By expressing the golden ratio as the sum of the twenty-fourth root of a fixed integer and a convergent exponential correction term, a highly accurate numerical approximation of φ has been achieved.

The proposed formulation demonstrates that the dominant contribution to the golden ratio can be effectively captured through a carefully selected root expression, while the residual error can be systematically corrected using the tail of an exponential series. This approach not only ensures rapid convergence but also provides a transparent step-by-step computational framework that aligns closely with the known numerical value of the golden ratio.

Beyond numerical accuracy, the significance of this work lies in its methodological contribution. The root–exponential hybrid structure introduced in this paper offers a new analytical perspective for representing irrational constants. Such constructions may be extended to explore alternative approximations of other mathematical constants, including π and e, and may find applications in numerical analysis, mathematical modeling, and theoretical investigations.

The results presented here reinforce the richness of the golden ratio as a mathematical object and demonstrate that even well-studied constants can yield new insights through innovative analytical formulations. It is hoped that this work will encourage further research into hybrid approximation techniques and their broader mathematical implications.

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