ACADEMIA International Journal for Social Sciences
Volume 4, Issue 4, 2025 ISSN-L (Online): 3006-6638

A Theoretical Study of Generalized Auxiliary Variable Estimators for Finite Population Mean under Measurement Error and Non-Response

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Received: 18-10-2025 **Revised:** 23-11-2025 **Accepted:** 23-12-2025 **Published:** 23-12-2025

ABSTRACT

This study proposes a generalized difference–cum–exponential type estimator for estimating the finite population mean under simple random sampling without replacement in the presence of measurement error and non-response. Auxiliary information is incorporated to improve estimation efficiency under a specific non-response scenario. Using first-order approximations, expressions for the bias and mean square error (MSE) of the proposed estimator are derived. The optimum values of the involved constants are obtained by minimizing the MSE. Theoretical efficiency conditions are established to assess the performance of the estimator. The proposed estimator is analytically compared with existing estimators, including those of Hansen and Hurwitz (1946), Cochran (1977), Rao (1986), Bahl and Tuteja (1991), and Kumar and Bhougal (2011). Results based on MSE comparisons show that the proposed estimator outperforms the competing estimators under realistic survey conditions. The findings indicate that the proposed estimator is a reliable and efficient alternative for practical survey applications affected by measurement error and non-response.

Keywords: *Auxiliary variable, bias, dual, exponential estimator, mean square error, non-sampling errors, measurement error, non-response.*

INTRODUCTION

Over the past thirty years, survey response rates have decreased. Accordingly, researchers should focus on increasing response rates and lowering the possibility of non-response mistakes. It was expected that unwilling sample members might supply data with measurement error and should be effectively persuaded to join the responder pool. In this case, two questions come up Makhdum, M. et al (2020). The first concerns the quality of a statistic (such as means or correlation coefficients) derived from a survey; that is, does the mean square error of a statistic rise when sample individuals who are corporate or less likely to be contacted are included in the respondent pool? Second, the topic relates to methodological investigations for identifying bias in non-response Azeem M and Hanif M. (2017).

Numerous academics have independently examined the characteristics of estimators when non-response and measurement errors are present. Measurement error has generally not been taken into account by researchers who have examined non-response, and vice versa. The characteristics of estimators for estimating the population mean of the research variable in the presence of non-response and measurement error were examined in this context in his doctoral thesis. When the population mean of the auxiliary variable is known, the work on estimating the population mean of the research variable in the presence of non-response and measurement error has been extended to estimate the parameters Azeem, M., & Hanif, M. (2015).

Several scholars have independently investigated the properties of estimators in the presence of measurement errors and non-response. Researchers that have studied non-response have typically not considered measurement error, and vice versa. In his PhD thesis, he looked at the properties of estimators for estimating the population mean of the study variable in the presence of measurement error and non-response Ismail, M., Shahbaz, M.S. and Hanif, M. (2011). The work on estimating the population mean of the research variable in the context of non-response and measurement error has been expanded to estimate the parameters when the population mean of the auxiliary variable is known Bahl S and Tuteja R (1991).

Using a straightforward random sample method without replacement, we have proposed a generalized estimator in the current work for the scenario of non-response and measurement errors on both study and auxiliary variables Bhushan, S., & Kumar, A. (2017).

FORMATION OF SAMPLES

A simple random sample of size n is selected from the population of size N using the simple random sampling without replacement (SRSWOR) method. Let be the observed values and the actual values for each of the two characteristics (x, y) associated with theith (i = 1,2,...,n) sample unit. Assume the measurement error of the study variable is Hansen, M.H. and Hurwitz, W.N. (1946)

$$U_i = y_i - Y_i$$

$$V_i = x_i - X_i$$

The measurement error are assumed to be random in nature and they are uncorrelated with mean zero and variances $S_v^2 = \frac{\sum_{i=1}^{N}(v_i - \bar{V})^2}{N_1 - 1}$ and $S_u^2 = \frac{\sum_{i=1}^{N}(u_i - \bar{U})^2}{N_1 - 1}$ respectively. It is also assumed that the true values of the variables Y and X are independent of measurement error.

In the event that there is some non-response, it is further assumed that the population of size N is divided into two mutually exclusive groups: respondents and non-respondents, the sizes of which are unknown Kumar, S. (2016).

Every sample unit falls into one of the three categories listed below:

1. They are the respondents;
2. They were not contacted again after completing the postal questionnaire; and
3. They are the respondents who answered when approached for a face-to-face interview.

We have Information from respondents' group and from subsampled respondent group.

Let (X_i^*, Y_i^*)be truevalues and (x_i^*, y_i^*)be the observed values of two characteristics (x,y)respectively linked with the $i=1,2,\ldots\ldots, n$ sample unit. Let the measurement error associated with the study variable in presence of non-response be

$$U_i^*=y_i^*-Y_i^*$$

If there is some non-response and measurement error present on the auxiliary variable then let

$$V_i^*=x_i^*-X_i^*$$

The measurement error are random in nature having mean zero and variances $S_{v_2}^2=\frac{\sum_{i=1}^{N}(v_i-\overline{V})^2}{N_2-1}$and $S_{u_2}^2=\frac{\sum_{i=1}^{N}(u_i-\overline{U})^2}{N_2-1}$. Let $S_{x_2}^2=\frac{\sum_{i=1}^{N_2}(x_i-\overline{X})^2}{N_2-1}$and $S_{y_2}^2=\frac{\sum_{i=1}^{N_2}(y_i-\overline{Y})^2}{N_2-1}$ be the variances for variables *X* and *Y* respectively for the sub sampled respondent's population It is further assumed that the measurement error for variables *Y* and *X* are independent.

ASSUMPTIONS

The following presumptions apply to measurement inaccuracy and non-response:

1. In accordance with Hansen and Hurwitz (1946), we make the assumption that the subsample group has a full response. This assumption is based on the fact that the personal interview approach has a lower non-response rate.

2. We also made the assumption that the measurement error means were zero.

3. In accordance with Singh and Karpe (2009), we made the assumption that there is no measurement error in the variables' real values.

A FEW CURRENT ESTIMATORS

Hansen and Hurwitz (1946) suggested the mean estimator in the presence of non-response and measurement error is given by

$$\overline{y}^*=\frac{n_1}{n}\overline{y}_{n_1}^*+\frac{n_2}{n}\overline{y}_r^*$$

Variance of estimator is given as:

$$MSE\left(\overline{y}^*\right)=\lambda_2\left(S_Y^2+S_U^2\right)+\vartheta\left(S_{Y_2}^2+S_{U_2}^2\right)$$

The ratio and product type estimator of population mean was created by Cochran (1977). When measurement error and non-response are present, the estimate is provided as:

$$t_r=\frac{\overline{y}^*}{\overline{x}^*}\overline{X};\ \ t_p=\frac{\overline{y}^*}{\overline{X}}\overline{x}^*$$

Mean square error of above mention estimators in the presence of measurement error and non-response is given as

$$MSE(t_r)\approx\lambda_2\left(S_Y^2+R^2S_X^2-2R\rho_{YX}S_YS_X\right)+\lambda_2\left(S_U^2+R^2S_V^2\right)+\vartheta\left(S_{U_2}^2+R^2S_{V_2}^2\right)+\vartheta\left(S_{Y_2}^2+R^2S_{X_2}^2-2R\rho_{YX2}S_{Y_2}S_{X_2}\right)$$

And

$$MSE(t_p)\approx\lambda_2\left(S_Y^2+R^2S_X^2+2R\rho_{YX}S_YS_X\right)+\lambda_2\left(S_U^2+R^2S_V^2\right)+\vartheta\left(S_{U_2}^2+R^2S_{V_2}^2\right)+\vartheta\left(S_{Y_2}^2+R^2S_{X_2}^2+2R\rho_{YX2}S_{Y_2}S_{X_2}\right)$$

Regression estimator under measurement error and non-response

$t_{reg}=\bar{y}^*+b\left(\bar{X}-\bar{x}^*\right)$

$b=\frac{S_{xy}}{S_x^2}$And the S_{xy} , S_x^2 are based on information under the given sampling design. The minimumMSE of regression estimator is given as

$$min.MSE(t_{reg}) =\lambda_2\left(S_Y^2+S_U^2\right)+\vartheta\left(S_{U(2)}^2+S_{Y(2)}^2\right) +b_0^2\left(\lambda_2\left(S_x^2+S_v^2\right)+\vartheta\left(S_{x(2)}^2+S_{v(2)}^2\right)\right)-2b_0\left(\lambda_2\rho_{YX}S_YS_X-2\vartheta\rho_{YX2}S_{Y_2}S_{X_2}\right),$$

$$b_o=\frac{\lambda_2\rho_{YX}S_YS_X-2\vartheta\rho_{YX2}S_{Y_2}S_{X_2}}{\lambda_2\left(S_x^2+S_v^2\right)+\vartheta\left(S_{V_2}^2+S_{X_2}^2\right)}$$

Regression estimator under measurement error is given as

$t_{reg}=\bar{y}+b(\bar{X}-\bar{x})$

The minimums of regression estimator is given as

$$min.MSE(t_{reg})=\lambda_2\left(S_Y^2+S_U^2\right) +b_0^2\left(\lambda_2\left(S_x^2+S_v^2\right)\right)-2b_0\left(\lambda_2\rho_{YX}S_YS_X\right)$$

Where $b_o=\frac{\lambda_2\rho_{YX}S_YS_X}{\lambda_2\left(S_x^2+S_v^2\right)}$

Regression estimator under non-response on study variable and measurement error on both variables is given as

$t_{reg}=\bar{y}^*+b(\bar{X}-\bar{x})$

The minimum MSE of regression estimator is given as

$$min.MSE(t_{reg})=\lambda_2\left(S_Y^2+S_U^2\right)+\vartheta\left(S_{U(2)}^2+S_{Y(2)}^2\right) +b_0^2\left(\lambda_2\left(S_x^2+S_v^2\right)\right)-2b_0\lambda_2\rho_{YX}S_YS_X$$

Where $b_o=\frac{\lambda_2\rho_{YX}S_YS_X}{\lambda_2\left(S_x^2+S_v^2\right)}$

Rao (1986) developed following estimators

$$t_{R(a)}=\frac{\bar{y}^*}{\bar{x}}\bar{X},\quad t_{R(b)}=\frac{\bar{y}^*}{\bar{X}}\bar{x}$$

If measurement error and non-response is taken in account the mean square error of $t_{R(a)}$ is given as

$$MSE(t_{R(a)}) \approx \lambda_2 (S_Y^2 + R^2 S_X^2 - 2R\rho_{YX} S_Y S_X) + \lambda_2 (S_U^2 + R^2 S_V^2) + \vartheta (S_{U_2}^2 + S_{Y_2}^2)$$

And

$$MSE(t_{R(b)}) \approx \lambda_2 (S_Y^2 + R^2 S_X^2 + 2R\rho_{YX} S_Y S_X) + \lambda_2 (S_U^2 + R^2 S_V^2) + \vartheta (S_{U_2}^2 + S_{Y_2}^2)$$

Rao (1991) suggested a difference-type estimator for population mean in the presence of measurement error and non-response is defined as:

$$T_{rd} = \omega_1 (\bar{X} - \bar{x}^*) + \omega_2 \bar{y}^*$$

Where ω_1 and ω_2 are suitable chosen constants

$$.MSE(T_{rd}) \approx \bar{Y}^2 - 2\omega_2 \bar{Y}^2 + \omega_2^2 \bar{Y}^2 + \omega_1^2 \left(\lambda_2 (S_x^2 + S_v^2) + \vartheta (S_{X_2}^2 + S_{V_2}^2)\right) + \omega_2^2 \left(\lambda_2 (S_Y^2 + S_U^2) + \vartheta (S_{U_2}^2 + S_{Y_2}^2)\right) - 2\omega_1 \omega_2 \left(\lambda_2 \rho_{YX} S_Y S_X + \vartheta \rho_{YX_2} S_{Y_2} S_{X_2}\right)$$

Where as

$$\omega_1 = \frac{\bar{Y}^2 \left(\lambda_2 \rho_{YX} S_Y S_X - 2\vartheta \rho_{YX_2} S_{Y_2} S_{X_2}\right)}{\left(\lambda_2 (S_x^2 + S_v^2) + \vartheta (S_{X_2}^2 + S_{V_2}^2)\right)\left(\bar{Y}^2 + \lambda_2 (S_Y^2 + S_U^2) + \vartheta (S_{U_2}^2 + S_{Y_2}^2)\right) - \left(\lambda_2 \rho_{YX} S_Y S_X + \vartheta \rho_{YX_2} S_{Y_2} S_{X_2}\right)^2};$$

$$\omega = \frac{\bar{Y}^2 \left(\lambda_2 (S_x^2 + S_v^2) + \vartheta (S_{X_2}^2 + S_{V_2}^2)\right)}{\left(\lambda_2 (S_x^2 + S_v^2) + \vartheta (S_{X_2}^2 + S_{V_2}^2)\right)\left(\bar{Y}^2 + \lambda_2 (S_Y^2 + S_U^2) + \vartheta (S_{U_2}^2 + S_{Y_2}^2)\right) - \left(\lambda_2 \rho_{YX} S_Y S_X + \vartheta \rho_{YX_2} S_{Y_2} S_{X_2}\right)^2}$$

T_{Rd} Is improvement over conventional difference-type estimator given as

$$t_d = \bar{y}^* + k_1 (\bar{X} - \bar{x}^*)$$

k_1 Is suitably chosen constant such that MSE of estimator is minimum

$$min.MSE(t_d) \approx \lambda_2 (S_Y^2 + S_U^2) + \vartheta (S_{U_2}^2 + S_{Y_2}^2) + {k_{1o}}^2 \left(\lambda_2 (S_x^2 + S_v^2) + \vartheta (S_{X_2}^2 + S_{V_2}^2)\right) - 2k_{1o} \left(\lambda_2 \rho_{YX} S_Y S_X - 2\vartheta \rho_{YX_2} S_{Y_2} S_{X_2}\right)$$

$$\therefore k_{1o} = \frac{\lambda_2 \rho_{YX} S_Y S_X - 2\vartheta \rho_{YX_2} S_{Y_2} S_{X_2}}{\lambda_2 (S_x^2 + S_v^2) + \vartheta (S_{X_2}^2 + S_{V_2}^2)}$$

Bahl and Tuteja (1991) suggested a ratio and product –cum-exponential estimator of population mean and below is the expression in presence of non-response and measurement error and its mean square error.

$$t_{re} = \bar{y}^* exp\left(\frac{\bar{X} - \bar{x}^*}{\bar{X} + \bar{x}^*}\right); t_{pe} = \bar{y}^* exp\left(\frac{\bar{x}^* - \bar{X}}{\bar{X} + \bar{x}^*}\right)$$

$$MSE(t_{re}) \approx \lambda_2 \left[S_Y^2 + \frac{1}{4} R^2 S_X^2 - R\rho_{YX} S_Y S_X\right] + \vartheta \left[S_{Y_2}^2 + \frac{1}{4} R^2 S_{X_2}^2 - R\rho_{YX2} S_{Y_2} S_{X_2}\right] + \lambda_2 \left[S_U^2 + \frac{1}{4} R^2 S_V^2\right] + \vartheta \left[S_{U_2}^2 + \frac{1}{4} R^2 S_{V_2}^2\right]$$

$$MSE(t_{pe}) \approx \lambda_2 \left[S_Y^2 + \frac{1}{4} R^2 S_X^2 + R\rho_{YX} S_Y S_X\right] + \vartheta \left[S_{Y_2}^2 + \frac{1}{4} R^2 S_{X_2}^2 + R\rho_{YX2} S_{Y_2} S_{X_2}\right] + \lambda_2 \left[S_U^2 + \frac{1}{4} R^2 S_V^2\right] + \vartheta \left[S_{U_2}^2 + \frac{1}{4} R^2 S_{V_2}^2\right]$$

Kumar and Bhougal (2011) suggested the following exponential-type estimator of population mean in the presence of measurement error and non-response.

$$t_{kb}=\bar{y}^{*}\left\{\alpha\, exp\left(\frac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}\right)+(1-\alpha)\,exp\left(\frac{\bar{x}^{*}-\bar{X}}{\bar{X}+\bar{x}^{*}}\right)\right\}$$

Where α is a real constant such that the mean square error of t_{kb} is minimum.

$$min.MSE(t_{kb})\approx\lambda_2\left[S_Y^2+\left(\alpha_o-\frac{1}{2}\right)^2R^2S_X^2-2\left(\alpha_o-\frac{1}{2}\right)R\rho_{YX}S_YS_X\right]+\vartheta\left[S_{Y_2}^2+\left(\alpha_o-\frac{1}{2}\right)^2R^2S_{X_2}^2-2\left(\alpha_o-\frac{1}{2}\right)R\rho_{YX2}S_{Y_2}S_{X_2}\right]+\lambda_2\left[S_U^2+\left(\alpha_o-\frac{1}{2}\right)^2R^2S_V^2\right]+\vartheta\left[S_{U_2}^2+\left(\alpha_o-\frac{1}{2}\right)^2R^2S_{V_2}^2\right]$$

SUGGESTED ESTIMATORS

In the event of measurement error and non-response on the study and auxiliary variables, we have suggested a modified generalized difference cum exponential type estimator Rao T (1991).

$$t_{sm}^{*}=\left[\bar{y}^{*}\left\{\alpha\ exp\frac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}+(1-\alpha)\ exp\frac{\bar{x}^{*}-\bar{X}}{\bar{X}+\bar{x}^{*}}\right\}+\eta_1\left(\bar{X}-\bar{x}^{*}\right)+\eta_2\bar{y}^{*}\right]\left[\delta exp\frac{\bar{X}-\bar{x}^{*}}{\bar{X}+\bar{x}^{*}}+(1-\delta)\ exp\frac{\bar{x}^{*}-\bar{X}}{\bar{X}+\bar{x}^{*}}\right] \quad (5.1)$$

Where (α, δ) are constants such that $0 \le (\alpha, \delta) \le 1$ and (η_1,η_2) are constants such that MSE of the proposed class of estimators t_{sm}^{*} is minimum.

In order to obtain the mean square error of our suggested estimator let introduced some notations

$$\omega_Y^{*}=\sum_{i=1}^{n}\left(Y_i^{*}-\bar{Y}\right) \quad (5.2)$$

$$\omega_U^{*}=\sum_{i=1}^{n}U_i^{*} \quad (5.3)$$

$$\omega_X^{*}=\sum_{i=1}^{n}\left(X_i^{*}-\bar{X}\right) \quad (5.4)$$

And

$$\omega_V^{*}=\sum_{i=1}^{n}V_i^{*} \quad (5.5)$$

Adding *(5.2)* and *(5.3)*

$$\omega_Y^{*}+\omega_U^{*}=\sum_{i=1}^{n}\left(Y_i^{*}-\bar{Y}\right)+\sum_{i=1}^{n}U_i^{*}$$

By dividing both sides by *n*, we get

$$\frac{1}{n}\left(\omega_Y^{*}+\omega_U^{*}\right)=\frac{1}{n}\left[\sum_{i=1}^{n}\left(Y_i^{*}-\bar{Y}\right)+\sum_{i=1}^{n}U_i^{*}\right]$$

By simplifying we get

$$\bar{y}^{*}=\bar{Y}+\frac{1}{n}\left(\omega_Y^{*}+\omega_U^{*}\right) \quad (5.6)$$

Similarly, from *(5.4)* and *(5.5)*, we get

$$\bar{x}^{*}=\bar{X}+\frac{1}{n}\left(\omega_X^{*}+\omega_V^{*}\right) \quad (5.7)$$

Further

$$\left.\begin{aligned} E\left(\frac{1}{n}\left(\omega_Y^{*}+\omega_U^{*}\right)\right)^2 &= \lambda_2\left(S_Y^2+S_U^2\right)+\vartheta\left(S_{Y_2}^2+S_{U_2}^2\right) \\ E\left(\frac{1}{n}\left(\omega_X^{*}+\omega_V^{*}\right)\right)^2 &= \lambda_2\left(S_X^2+S_V^2\right)+\vartheta\left(S_{X_2}^2+S_{V_2}^2\right) \\ E\left[\frac{1}{n}\left(\omega_Y^{*}+\omega_U^{*}\right).\frac{1}{n}\left(\omega_X^{*}+\omega_V^{*}\right)\right] &= \lambda_2\rho_{YX}S_YS_X+\vartheta\rho_{XY_2}S_{Y_2}S_{X_2} \end{aligned}\right\} \quad (5.8)$$

Using *(5.6)* and *(5.7)* in *(5.1)* we get

$$t_{sm}^{*}=\left[\bar{y}^{*}\left\{\alpha\exp\frac{\bar{X}-\bar{X}-\frac{(\omega_X^{*}+\omega_V^{*})}{n}}{\bar{X}+\bar{X}+\frac{(\omega_X^{*}+\omega_V^{*})}{n}}+(1-\alpha)\exp\frac{\bar{X}-\bar{X}+\frac{(\omega_X^{*}+\omega_V^{*})}{n}}{\bar{X}+\bar{X}+\frac{(\omega_X^{*}+\omega_V^{*})}{n}}\right\}+\eta_1\left(\bar{X}-\bar{X}-\frac{(\omega_X^{*}+\omega_V^{*})}{n}\right)+\eta_2\bar{Y}+\frac{\eta_2}{n}\left(\omega_Y^{*}+\omega_U^{*}\right)\right]\left[\delta\exp\frac{\bar{X}-\bar{X}-\frac{(\omega_X^{*}+\omega_V^{*})}{n}}{\bar{X}+\bar{X}+\frac{(\omega_X^{*}+\omega_V^{*})}{n}}+(1-\delta)\exp\frac{\bar{X}-\bar{X}+\frac{(\omega_X^{*}+\omega_V^{*})}{n}}{\bar{X}+\bar{X}+\frac{(\omega_X^{*}+\omega_V^{*})}{n}}\right]$$

By expanding the Taylor's series in above equation and by ignore the higher order of approximation we get:

$$t_{sm}^{*}=\left[\left(\bar{Y}+\frac{1}{n}\left(\omega_Y^{*}+\omega_U^{*}\right)\right)\left\{\alpha\exp\left\{\frac{-1}{2\bar{X}}\frac{(\omega_X^{*}+\omega_V^{*})}{n}+\frac{1}{4\bar{X}^2}\left(\frac{\omega_X^{*}+\omega_V^{*}}{n}\right)^2\right\}+(1-\alpha)\exp\left\{\frac{-1}{2\bar{X}}\frac{(\omega_X^{*}+\omega_V^{*})}{n}+\frac{1}{4\bar{X}^2}\left(\frac{\omega_X^{*}+\omega_V^{*}}{n}\right)^2\right\}\right\}-\eta_1\frac{(\omega_X^{*}+\omega_V^{*})}{n}+\eta_2\bar{Y}+\frac{\eta_2}{n}\left(\omega_Y^{*}+\omega_U^{*}\right)\right]$$

By applying the expectations, we get the Bias for the proposed estimator:

$$Bias\left(t_{sm}^{*}\right)=\bar{Y}\,(t_{sm}^{*}-\bar{Y})^2=\bar{Y}^2\left[\left(\frac{1}{\bar{X}}-\frac{\delta}{\bar{X}}-\frac{\alpha}{\bar{X}}\right)^2\left(\frac{\omega_X^{*}+\omega_V^{*}}{n}\right)^2+\left(\frac{\omega_Y^{*}+\omega_U^{*}}{n\bar{Y}}\right)^2+\eta_1^2\left(\frac{\omega_X^{*}+\omega_V^{*}}{n\bar{Y}}\right)^2+\eta_2^2\left\{1+\left(\frac{1}{2\bar{X}}-\frac{\delta}{\bar{X}}\right)^2\left(\frac{\omega_X^{*}+\omega_V^{*}}{n}\right)^2+\frac{1}{\bar{Y}^2}\left(\frac{\omega_Y^{*}+\omega_U^{*}}{n}\right)^2+\right.\right.$$
$$2\left(\frac{1}{2\bar{X}}-\frac{\delta}{\bar{X}}\right)\frac{(\omega_X^{*}+\omega_V^{*})}{n}+\frac{2}{\bar{Y}}\frac{(\omega_Y^{*}+\omega_U^{*})}{n}+2\left(\frac{1}{2\bar{X}}-\frac{\delta}{\bar{X}}\right)\frac{1}{\bar{Y}}\frac{(\omega_X^{*}+\omega_V^{*})}{n}\frac{(\omega_Y^{*}+\omega_U^{*})}{n}-2\left(\frac{1}{4}-\delta\right)\frac{1}{2\bar{X}^2}\left(\frac{\omega_Y^{*}+\omega_U^{*}}{n}\right)^2+2\left(\frac{1}{2\bar{X}}\frac{\delta}{\bar{X}}\right)\frac{1}{\bar{Y}}\frac{(\omega_X^{*}+\omega_V^{*})}{n}\frac{(\omega_Y^{*}+\omega_U^{*})}{n}\right\}+$$
$$\frac{2}{\bar{Y}}\left(\frac{1}{\bar{X}}-\frac{\delta}{\bar{X}}-\frac{\alpha}{\bar{X}}\right)\frac{(\omega_X^{*}+\omega_V^{*})}{n}\frac{(\omega_Y^{*}+\omega_U^{*})}{n}-\eta_1\frac{2}{\bar{Y}}\left(\frac{1}{\bar{X}}-\frac{\delta}{\bar{X}}-\frac{\alpha}{\bar{X}}\right)\left(\frac{\omega_X^{*}+\omega_V^{*}}{n}\right)^2-\eta_1\frac{2}{\bar{Y}^2}\frac{(\omega_X^{*}+\omega_V^{*})}{n}\frac{(\omega_Y^{*}+\omega_U^{*})}{n}+\eta_2 2\left\{\frac{1}{\bar{Y}^2}\left(\frac{\omega_Y^{*}+\omega_U^{*}}{n}\right)^2+\left(2\left(\frac{1}{\bar{X}}-\frac{\delta}{\bar{X}}-\frac{\alpha}{\bar{X}}\right)+\right.\right.$$
$$\left.\left(\frac{1}{2\bar{X}}-\frac{\delta}{\bar{X}}\right)\right)\frac{1}{\bar{Y}}\frac{(\omega_X^{*}+\omega_V^{*})}{n}\frac{(\omega_Y^{*}+\omega_U^{*})}{n}+\left(\left(\frac{1}{\bar{X}}-\frac{\delta}{\bar{X}}-\frac{\alpha}{\bar{X}}\right)\left(\frac{1}{2\bar{X}}-\frac{\delta}{\bar{X}}\right)+\frac{\delta\alpha}{\bar{X}^2}\right)\left(\frac{\omega_X^{*}+\omega_V^{*}}{n\bar{Y}}\right)^2\right\}-2\eta_1\eta_2\left\{\frac{1}{\bar{Y}}\frac{(\omega_X^{*}+\omega_V^{*})}{n}+\left(\frac{1}{2\bar{X}}-\frac{\delta}{\bar{X}}\right)\frac{1}{\bar{Y}}\left(\frac{\omega_X^{*}+\omega_V^{*}}{n}\right)^2+\frac{1}{\bar{Y}^2}\frac{(\omega_X^{*}+\omega_V^{*})}{n}\frac{(\omega_Y^{*}+\omega_U^{*})}{n}+\right.$$
$$\left.\left.\left(\frac{1}{2\bar{X}}-\frac{\delta}{\bar{X}}\right)\frac{1}{\bar{Y}}\left(\frac{\omega_X^{*}+\omega_V^{*}}{n}\right)^2\right\}\right]$$

Applying expectation on both side of equation we get MSE of proposed estimator

$$MSE(t_{sm}^{*})=\bar{Y}^2\left[A_0+\eta_1^2A_1+\eta_2^2A_2-2\eta_1\eta_2A_3-2\eta_1A_4+2\eta_2A_5\right] \quad (5.11)$$

Differentiating Equation *(5.11)* w.r.t η_1 and η_2 and equating them to zero, we get

$$\begin{bmatrix} A_1 & -A_3 \\ -A_3 & A_2 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}=\begin{bmatrix} A_4 \\ -A_5 \end{bmatrix}$$

After simplification of above equation, the optimum values of η_1 and η_2 is

$$\left.\begin{aligned}\eta_{11}&=\frac{A_2A_4\text{-}A_3A_5}{A_1A_2\text{-}A_3^2}\\ \eta_{12}&=\frac{A_3A_4\text{-}A_1A_5}{A_1A_2\text{-}A_3^2}\end{aligned}\right]$$

Substituting the value of inη_1 and η_2 equation *(5.11)*

$$.MSE(\mathrm{t}_{sm}^{*})\ =\bar{Y}^2\left[A_0+\eta_{11}^2A_1+\eta_{12}^2A_2\text{-}\ 2\eta_{11}\eta_{12}A_3\text{-}2\eta_{11}A_4+\quad 2\eta_{12}A_5\right]$$

or

$$min.MSE\left(\mathrm{t}_{sm}^{*}\right)=\bar{Y}^2\left[A_0\text{-}\frac{A_2A_4^2\ \text{-}\ 2A_3A_4A_5+A_1A_5^2}{\left(A_1A_2\ \text{-}\ A_3^2\right)}\right]$$

Whereas

$$A_0\ =\beta^2\lambda_2\left(S_X^2+S_V^2\right)+\beta^2\vartheta\left(S_{X_2}^2+S_{V_2}^2\right)+\ \frac{1}{\bar{Y}^2}\left(\lambda_2\left(S_Y^2+S_U^2\right)+\vartheta\left(S_{Y_2}^2+S_{U_2}^2\right)\right)+\frac{2}{\bar{Y}}\beta\left(\lambda_2\rho_{YX}S_YS_X+\vartheta\rho_{XY_2}S_{Y_2}S_{X_2}\right)$$

$$A_1=\frac{1}{\bar{Y}^2}\left\{\lambda_2\left(S_X^2+S_V^2\right)+\vartheta\left(S_{X_2}^2+S_{V_2}^2\right)\right\}$$

$$A_2=1+\left(\frac{\delta^2}{\bar{X}^2}\right)\left(\lambda_2\left(S_X^2+S_V^2\right)+\vartheta\left(S_{X_2}^2+S_{V_2}^2\right)\right)+\frac{1}{\bar{Y}^2}\left(\lambda_2\left(S_Y^2+S_U^2\right)+\vartheta\left(S_{Y_2}^2+\ S_{U_2}^2\right)\right)+4\tau\frac{1}{\bar{Y}}\left(\lambda_2\rho_{YX}S_YS_X+\vartheta\rho_{XY_2}S_{Y_2}S_{X_2}\right)$$

$$A_3=\left\{2\tau\ \frac{1}{\bar{Y}}\left(\lambda_2\left(S_X^2+S_V^2\right)+\vartheta\left(S_{X_2}^2+S_{V_2}^2\right)\right)+\frac{1}{\bar{Y}^2}\left(\lambda_2\rho_{YX}S_YS_X+\vartheta\rho_{XY_2}S_{Y_2}S_{X_2}\right)\right\}$$

$$A_4=\frac{1}{\bar{Y}}\beta\left(\lambda_2\left(S_X^2+S_V^2\right)+\vartheta\left(S_{X_2}^2+S_{V_2}^2\right)\right)+\frac{1}{\bar{Y}^2}\left(\lambda_2\rho_{YX}S_YS_X+\vartheta\rho_{XY_2}S_{Y_2}S_{X_2}\right)$$

EMPIRICAL RESEARCH

Using the R programming language, we created four distinct populations from a normal distribution with various parameters for the empirical investigation in this part. The population for Y and X in the simulation research is taken from the following populations and used as a multivariate normal Sabir, S., &Sanaullah, A. (2019).

To validate the results, a Monte-Carlo simulation study is conducted using the R software. The data matrix was created for 10,000 replications and two non-overlapping groups based on the various population sizes. The multivariate normal distribution for variables with mean vectors (,0,0) and covariance matrix was used to create the data matrix for X, Y, V, and U. Vishwakarma, G. K et al (2019).

$$=\begin{bmatrix} S_Y^2 & \rho_{XY}S_YS_X & 0 & 0\\ \rho_{XY}S_YS_X & S_X^2 & 0 & 0\\ 0 & 0 & S_U^2 & 0\\ 0 & 0 & 0 & S_V^2\end{bmatrix}$$

The populations are given below.

Population 1: (Cochran (1977.), p.196**)**

Let y be bushels in an orchard, x be the number of peach trees in the orchard, and y be the peach output in June 1946 in North Carolina. For this data collection, the statistics are:

N=256; n= 100; ρ=0.887

$$\mu=\begin{bmatrix}56.47\\44.45\end{bmatrix};\ \Sigma=\begin{bmatrix}6430.019 & 4426.185\\4426.185 & 3872.573\end{bmatrix}$$

Population 2:

N= 5000; n= 500; ρ= -0.18275

$$\mu=\begin{bmatrix}1.007391\\9.953722\end{bmatrix};\quad \Sigma=\begin{bmatrix}1.0310780 & -0.3740459\\-0.3740459 & 4.0629690\end{bmatrix}$$

INTERPRETATION

The PRE of a few current estimators is provided for populations in Tables 1(a) and 2(a) for various choices of h and non-response rate in the presence of measurement error and non-response on both study and auxiliary variables, as well as in the presence of measurement error and non-response on both variables and non-response on study variable only, respectively. The PRE of the suggested estimators for various values of (α,δ) for various values of h and non-response rate are shown in tables 1(b) and 2(b).

Table 1(a) and (b) concludes that

- The suggested class of estimator is more efficient over other existing estimators for almost all different choice of h and non-response rate except the
  Population 1 for the value of *h*=4 for population 2 for h=4 and non-response rate 10%.
- For the Population 1 the PREs decrease respectively with the increase in value of hand non-response rate.
- For Population 2, it shows for PRE's are decreases with the increase in value of h and non-response rate. Because of negative correlation between variables.
- It is observed that by increasing the value of h, and by increasing non response rate the proposed estimator becomes more efficient
- PRE of generalized difference cum exponential type estimator increases with the increase in value of α while keeping δ constant.
- For all fixed values of α PRE's of proposed estimators increases by increasing the value of δ till reaching at δ=0.75.
- The highest values of PREs are obtain at the following combination of (α, δ) = (0.25, 0.75), (0.50, 0.75), (0.75, 0.75),d at (1, 0.75).

CONCLUSION

Tables 1-2 (see Appendix) provide the percent relative efficiencies (PREs) of the proposed and several existing estimators with regard to the modified Hansen and Hurwitz (1946) estimator. In the presence of measurement error, non-response on both study and auxiliary variables, and a special case of the proposed estimator, the PREs have been computed for the modified generalized difference cum exponential type estimator of the finite unknown population mean that is proposed using auxiliary information. According to theoretical and simulated results, the estimator outperforms modified Hansen and Hurwitz (1964), Cochran (1977), Rao (1986), Bahl and Tuteja (1991), and Kumar and Bhougal (2011) generalized exponential estimators for all populations in the corresponding scenarios.

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APPENDIX

Table 1(a): PRE's of some modified existing similar type estimators in the presence of measurement error and non-response on both study and auxiliary variables

EST	NON-RESPONSE 10%				NON-RESPONSE 40%			
	Pop1		Pop2		Pop1		Pop2	
	***h*=2**	***h*=4**	***h*=2**	***h*=4**	***h*=2**	***h*=4**	***h*=2**	***h*=4**
$\bar{y}^*$	**100**	**100**	100	100	**100**	**100**	100	100
t_r	475.909	485.920	93.5407	93.3966	509.295	539.3301	93.4379	93.2387
t_p	25.4666	25.1608	101.634	101.835	26.0920	26.3703	101.762	102.028
t_{re}	295.894	302.692	97.285	97.2038	290.227	289.425	97.229	97.1177
t_{pe}	45.5322	45.1531	101.488	101.584	46.239	46.5359	101.551	101.679
t_{reg}	520.9153	538.463	101.740	101.903	539.383	562.6096	101.844	102.066
t_{rd}	522.340	540.189	102.038	102.338	541.532	566.5052	102.202	102.472

t_{kb}	62.0522	61.8704	100.004	100.003	61.8612	61.6404	100.004	100.004

Table 1(b):PRE's of the proposed modified generalized difference cum exponential type estimator t^*_{sm} in the presence of measurement error and non-response on both study and auxiliary variables

α	δ	NON-RESPONSE 10%				NON-RESPONSE 40%			
		Pop1		Pop2		Pop1		Pop2	
		***h*=2**	***h*=4**	***h*=2**	***h*=4**	***h*=2**	***h*=4**	***h*=2**	***h*=4**
0.25	0.25	525.906	544.781	102.132	102.376	**547.118**	**577.359**	102.313	102.771
	0.50	533.153	554.359	102.136	102.381	558.027	598.409	102.318	102.778
	0.75	**536.336**	**558.611**	**102.137**	**102.382**	**562.834**	**607.904**	**102.319**	**102.781**
	1	533.201	554.436	102.136	102.381	558.132	598.773	102.318	102.778
0.50	0.25	523.518	541.709	102.139	102.384	543.281	569.777	102.321	102.784
	0.50	529.142	549.100	102.143	102.389	551.587	585.400	102.326	102.792
	0.75	**531.752**	**552.570**	**102.144**	**102.391**	**555.458**	**592.818**	**102.328**	**102.794**
	1	529.17	549.151	102.143	102.389	551.655	585.627	102.326	102.792
0. 75	0.25	521.907	539.662	102.146	102.393	540.762	564.958	102.330	102.798
	0.50	525.953	544.941	102.150	102.398	546.575	575.564	102.335	102.805
	0.75	**528.014**	**547.6687**	**102.151**	**102.400**	**549.563**	**581.122**	**102.336**	**102.808**
	1	525.972	544.971	102.150	102.398	546.615	575.692	102.335	102.805
1	0.25	521.060	538.616	102.152	102.402	539.526	568.613	102.339	102.811

	0.50	523.558	541.834	102.15	102.407	542.91	568.613	102.344	102.819
	0.75	**525.088**	**543.848**	**102.158**	**102.409**	**545.062**	**572.4778**	**102.345**	**102.821**
	1	523.568	541.850	102.157	102.407	545.062	568.6711	102.344	102.819